Algorytmy kombinatoryczne i graficzne w spektralnej klasyfikacji skończonych bigrafów oraz sieciowych systemów pierwiastków

Combinatorial and graphical algorithms in the spectral classification of finite bigraphs and mesh root systems

Summary

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The aim of this dissertation is the classification up to the Gram congruences $\sim_{\mathbb{Z}}$ and $\approx_{\mathbb{Z}}$ and Coxeter spectral classification of non-negative loop-free edge-bipartite graphs (*bigraphs*) $\Delta = (\Delta_0, \Delta_1)$ (defined in [22]) with finite set of vertices $\Delta_0 = \{a_1, \dots a_m\}$ and a finite set of edges Δ_1 (labelled with symbols from the two-element set $\{-1, +1\}$), as well as the construction of algorithmic tools to perform this classification.

In particular, we present combinatorial and graphical algorithms allowing Gram classification and Coxeter spectral classification of loop-free edge-bipartite graphs of two corank two.

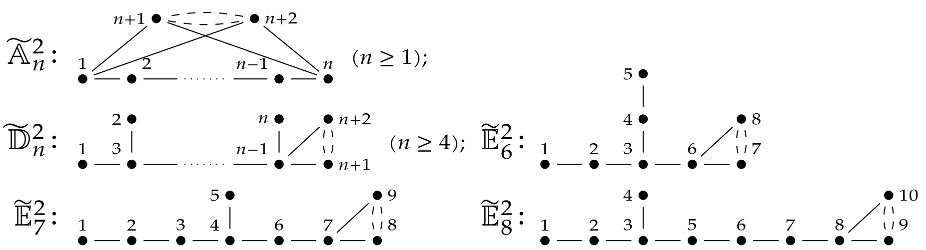
Main results and their applications

The main result of the classification up to the Gram congruences $\sim_{\mathbb{Z}}$ and $\approx_{\mathbb{Z}}$ and Coxeter spectral classification presented in this dissertation is a classification of non-negative loop-free edge-bipartite graphs of corank two. We say that a bigraph $\Delta = (\Delta_0, \Delta_1)$ with $m \geq 1$ vertices is non-negative of corank $0 \leq r \leq m-1$, if the symmetric Gram matrix $G_\Delta := \frac{1}{2}[\check{G}_\Delta + \check{G}_\Delta^{tr}] \in \mathbb{M}_m(\mathbb{Q})$ is positive semidefinite of rank $m - r$, where $\check{G}_\Delta \in \mathbb{M}_m(\mathbb{Z})$ is the non-symmetric Gram matrix viewed as a modified signed graph adjacency matrix.

non-negative loop-free edge-bipartite graphs Δ with $m \geq 1$ vertices are studied up to two Gram $\mathbb{Z}$-congruences: weak $\Delta \sim_{\mathbb{Z}} \Delta'$ and strong $\Delta \approx_{\mathbb{Z}} \Delta'$, where

$$\begin{aligned} \Delta \sim_{\mathbb{Z}} \Delta' \quad &\Leftrightarrow \quad G_{\Delta'} = B^{tr} \cdot G_\Delta \cdot B, \quad \text{for some } B \in \mathbb{M}_m(\mathbb{Z}),\ \det B = \pm 1, \\ \Delta \approx_{\mathbb{Z}} \Delta' \quad &\Leftrightarrow \quad \check{G}_{\Delta'} = B^{tr} \cdot \check{G}_\Delta \cdot B, \quad \text{for some } B \in \mathbb{M}_m(\mathbb{Z}),\ \det B = \pm 1. \end{aligned}$$

One of the most important results of this dissertation is the construction of the following family of extended Euclidean bigraphs $\widetilde{\mathbb{A}}_n^2, n \geq 1, \widetilde{\mathbb{D}}_n^2, n \geq 4, \widetilde{\mathbb{E}}_6^2, \widetilde{\mathbb{E}}_7^2, \widetilde{\mathbb{E}}_8^2$:

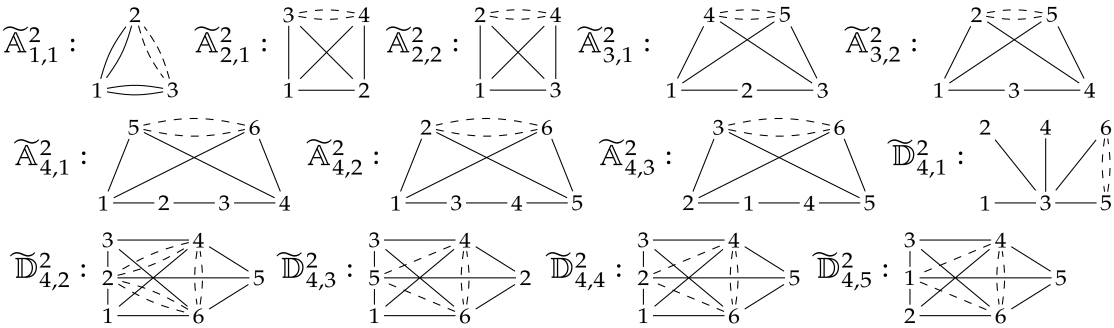


that classify all connected non-negative loop-free edge-bipartite graphs up to the weak Gram $\mathbb{Z}$-congruence $\sim_{\mathbb{Z}}$.

The main classification results of this dissertation are as follows.

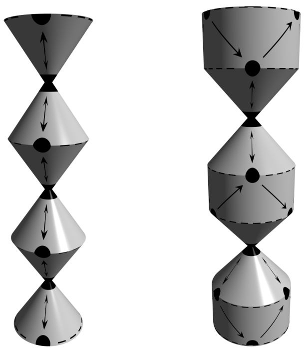
(I) We prove that, up to the weak Gram $\mathbb{Z}$-congruence $\sim_{\mathbb{Z}}$, every connected loop-free non-negative edge-bipartite graph of corank two is one of the extended Euclidean bigraphs $\widetilde{\mathbb{A}}^2_n, n \geq 1, \widetilde{\mathbb{D}}^2_n, n \geq 4, \widetilde{\mathbb{E}}^2_6, \widetilde{\mathbb{E}}^2_7, \widetilde{\mathbb{E}}^2_8$.

(II) We show that, up to the strong Gram $\mathbb{Z}$-congruence $\approx_{\mathbb{Z}}$, every connected loop-free non-negative edge-bipartite graph of corank two with at most six vertices is one of the following 13 bigraphs $\widetilde{\mathbb{A}}^2_{1,1}, \widetilde{\mathbb{A}}^2_{2,1}, \widetilde{\mathbb{A}}^2_{2,2}, \widetilde{\mathbb{A}}^2_{3,1}, \widetilde{\mathbb{A}}^2_{3,2}, \widetilde{\mathbb{A}}^2_{4,1}, \widetilde{\mathbb{A}}^2_{4,2}, \widetilde{\mathbb{A}}^2_{4,3}, \widetilde{\mathbb{D}}^2_{4,1}, \widetilde{\mathbb{D}}^2_{4,2}, \widetilde{\mathbb{D}}^2_{4,3}, \widetilde{\mathbb{D}}^2_{4,4}, \widetilde{\mathbb{D}}^2_{4,5}$:

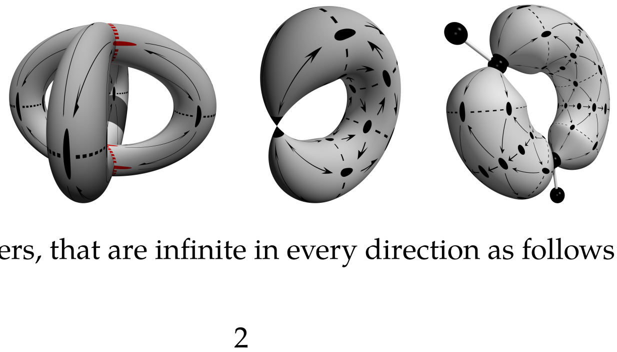


(III) For bigraphs with a small number of vertices we reduce the classification (up to the strong Gram $\mathbb{Z}$-congruence) of connected loop-free bigraphs of corank two to the classification of Φ_Δ-mesh geometry of roots (up to the isomorphism of $\Phi_\Delta, \Phi_{\Delta'}$-quivers, i.e. digraphs with "additional structure"). In a case of connected non-negative edge-bipartite graphs Δ of corank two without loops with at most $n+2 \leq 6$ vertices we construct the infinite set $\mathcal{R} \subset \mathbb{Z}^{n+2}$ consisting of Φ_Δ-orbits of all roots $\mathcal{R}_\Delta := \{v \in \mathbb{Z}^{n+2};\ v \cdot G_\Delta \cdot v^{tr} = 1\}$ of bigraph Δ and Φ_Δ-orbits of certain vectors from the kernel Ker $q_\Delta := \{v \in \mathbb{Z}^{n+2};\ v \cdot G_\Delta \cdot v^{tr} = 0\}$ with a structure of the Φ_Δ-mesh geometry. It is an infinite sum of:

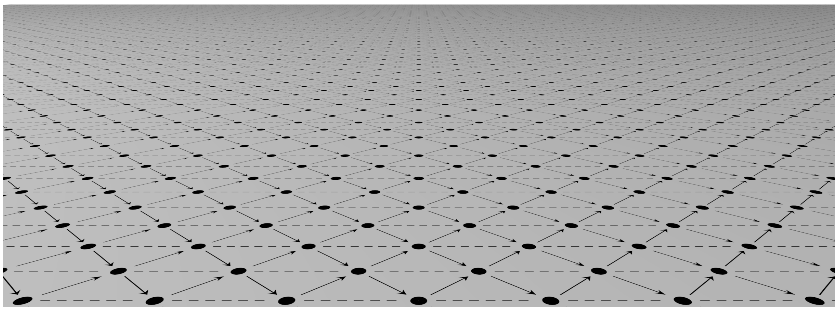
(i) infinite (up and down) rank 2 and 3 sand-glass tubes:



(ii) finite tori of rank $3, 4, 5$ and 6, including the following glued torus:



(iii) planar quivers, that are infinite in every direction as follows:



(IV) We build an algorithm (so-called inflation algorithm), that calculates in a polynomial time some matrix $B \in \mathbb{M}_{n+2}(\mathbb{Z})$ that defines weak Gram $\mathbb{Z}$-congruence $\Delta \sim_{\mathbb{Z}} \widetilde{D}_n^2$ between a connected loop-free non-negative bigraph $\Delta = (\Delta_0, \Delta_1)$ of corank two and $n+2 \geq 3$ vertices of Dynkin type D_n, and extended Euclidean bigraph $\widetilde{D}_n^2 \in \{\widetilde{\mathbb{A}}_n^2, n \geq 1, \widetilde{\mathbb{D}}_n^2, n \geq 4, \widetilde{\mathbb{E}}_6^2, \widetilde{\mathbb{E}}_7^2, \widetilde{\mathbb{E}}_8^2\}$. This matrix is described by the composite inflation operator $\mathbf{t}_\bullet^- := \mathbf{t}_{a_k b_k}^- \circ \ldots \circ \mathbf{t}_{a_1 b_1}^-$ the reduces a bigraph Δ to the bigraph $\widetilde{\mathbb{D}}_n^2$:

$$\Delta \mapsto \mathbf{t}_{a_1 b_1}^- \Delta \mapsto \mathbf{t}_{a_2 b_2}^- (\mathbf{t}_{a_1 b_1}^- \Delta) \mapsto \ \cdots \ \mapsto \mathbf{t}_\bullet^- \Delta = \widetilde{D}_n^2.$$

(V) We construct efficient combinatorial and graphical algorithms calculating the matrix defining the strong Gram $\mathbb{Z}$-congruence between connected loop-free non-negative eddge-bipartite graphs Δ, Δ' of corank two and at most six vertices.

One of the results of this dissertation is the package of algorithms suitable for combinatorial analysis of bigraphs by means of computational tools. Our implementations of the algorithms described in the dissertation can be used for further experimental research, i.e. the verification of complex hypotheses for which theoretical proofs are not known, as well as for computer-assisted proofs. As a consequence, we extend by new algorithms the available computational tools for analysing problems in spectral classification of bigraphs.

The results of the dissertation have a significant applications in the Coxeter spectral analysis of connected loop-free non-negative edge-bipartite graphs $\Delta = (\Delta_0, \Delta_1)$. In particular:

(a) classification of certain classes of non-negative integer unit quadratic forms can be reduced to the classification of connected loop-free non-negative edge-bipartite graphs of corank two

(b) our tools and combinatorial and graphical algorithms:
  - reduce the problem of existence of the strong Gram $\mathbb{Z}$-congruence between connected loop-free non-negative bigraphs of corank two with at most 6 vertices to the existence of the isomorphism between Φ-mesh geometries of roots;
  - build combinatorial and graphical description of integer solutions of certain quadratic Diophantine equations $q(x_1, \ldots, x_m) = d$ (in connection with X Hilbert's problem, cf. [20, 21]);

(c) presented results of Coxeter spectral analysis of non-negative loop-free edge-bipartite graphs have applications in the Coxeter spectral analysis of non-negative partially ordered sets and their matrix representations, see [9, 10, 24].

Motivation

One of the inspirations for the study of spectral invariants of edge-bipartite graphs (discussed extensively in the articles [20–22]) were the problems of Coxeter spectral classification of finite dimension algebras over a field K and their relationship with the so-called derive equivalence of algebras studied since the early 1980s, see the works of Gabriel-Roiter [5], Lenzing-Peña [16], Mróz [18], Mróz-Peña [19] and Simson [21].

Another important inspiration were problems close to the Hilbert X problem: construction of algorithms (preferably graphical ones) that describe geometrically the set of all integer solutions $v = (v_1, \dots, v_m) \in \mathbb{Z}^m$ of Diophantine equations $q(x_1, \dots, x_m) = d$, where $d \in \mathbb{Z}$ is an integer and $q(x_1, \dots, x_m) \in \mathbb{Z}[x_1, \dots, x_m]$ is a unit integer quadratic form, see [20, 21]. These problems are intensively studied by many authors, see [1, 2, 9, 10, 14, 15, 19, 20, 23, 24, 26].

A very important area of inspiration for this research are also classical problems and methods of spectral theory of graphs and finite signed graphs. They are used, among others, to describe and study various processes occurring in nature, the analysis of electrical networks, and even the analysis of phenomena studied in social sociology, including conflicts of social groups, see [3, 4, 13, 27].

We note that the results presented in the dissertation may have applications (beyond the main subject area discussed in the dissertation) in the classification of derived categories $\mathcal{D}^b(\mathrm{mod}\, R)$. In particular, in the study of the dependencies of the tubular structure of Auslander-Reiten quivers (directed graphs) in the relation to the cyclotomic factors of the Coxeter polynomials $\mathrm{cox}_R(t) \in \mathbb{Z}[t]$. The results of this type are presented in [18, 19, 21].

The main results presented in this dissertation have been published in the following scientific journals:

- *Linear Algebra and its Applications* [10, 11, 25],
- *European Journal of Combinatorics* [9],
- *Fundamenta Informaticae* [12, 26],
- *Algebra and Discrete Mathematics* [8],
- *International Journal of Mathematics and Mathematical Sciences* [24].

The results of the dissertation were presented at the following international scientific conferences and published in peer-reviewed conference materials:

- *Combinatorics 2012*, Perugia [23],
- *International Symposium on Symbolic and Numeric Algorithms for Scientific Computing*, Timișoara, SYNASC 2012 [6], SYNASC 2013 [17] oraz SYNASC 2014 [7].

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References

[1] M. Barot and J. A. de la Peña, *The Dynkin type of a non-negative unit form*, in: Exposition. Math. **17** (1999), 339–348.

[2] R. Bocian, M. Felisiak, and D. Simson, *Numeric and mesh algorithms for the Coxeter spectral study of positive edge-bipartite graphs and their isotropy groups*, in: J. Comput. Appl. Math. **259** (2014), 815–827, doi: 10.1016/j.cam.2013.07.013.

[3] D. Cvetković, P. Rowlinson, and S. Simić, "Eigenspaces of Graphs", vol. 66, Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, 1997, xiv+258 pp., doi: 10.1017/CBO9781139086547.

[4] D. Cvetković, P. Rowlinson, and S. Simić, "An Introduction to the Theory of Graph Spectra", London Math. Soc. Student Texts 75, Cambridge: Cambridge University Press, 2010, xii+364 pp., doi: 10.1017/CBO9780511801518.

[5] P. Gabriel and A. V. Roĭter, "Representations of Finite-Dimensional Algebras", Encyclopaedia of Mathematical Sciences 73, Springer, Berlin, 1992, iv+177 pp.,

[6] M. Gąsiorek, D. Simson, and K. Zając, "On Coxeter spectral study of posets and a digraph isomorphism problem", in: *Symbolic and Numeric Algorithms for Scientific Computing, SYNASC 2012*, IEEE CPS, Los Alamitos, Calif. [etc.], 2012, 369–375, doi: 10.1109/SYNASC.2012.56.

[7] M. Gąsiorek, D. Simson, and K. Zając, "On corank two edge-bipartite graphs and simply extended Euclidean diagrams", in: *Symbolic and Numeric Algorithms for Scientific Computing, SYNASC 2014*, IEEE CPS, Los Alamitos, Calif. [etc.], 2014, 66–73, doi: 10.1109/SYNASC.2014.17.

[8] M. Gąsiorek, D. Simson, and K. Zając, *Algorithmic computation of principal posets using Maple and Python*, in: Algebra and Discr. Math. **17** (2014), 33–69, URL: http://adm.luguniv.edu.ua/downloads/issues/2014/N1/adm-n1(2014)-4.pdf.

[9] M. Gąsiorek, D. Simson, and K. Zając, *On Coxeter type study of non-negative posets using matrix morsifications and isotropy groups of Dynkin and Euclidean diagrams*, in: European J. Combin. **48** (2015), 127–142, doi: 10.1016/j.ejc.2015.02.015.

[10] M. Gąsiorek, D. Simson, and K. Zając, *Structure and a Coxeter-Dynkin type classification of corank two non-negative posets*, in: Linear Algebra Appl. **469** (2015), 76–113, doi: 10.1016/j.laa.2014.11.003.

[11] M. Gąsiorek, D. Simson, and K. Zając, *A Gram classification of non-negative corank-two loop-free edge-bipartite graphs*, in: Linear Algebra Appl. **500** (2016), 88–118, doi: 10.1016/j.laa.2016.03.007.

[12] M. Gąsiorek and K. Zając, *On algorithmic study of non-negsative posets of corank at most two and their Coxeter-Dynkin types*, in: Fundamenta Informaticae **139** (2015), 347–367, doi: 10.3233/FI-2015-1238.

[13] F. Harary, *On the notion of balance of a signed graph*, in: Michigan Math. J **2** (1953), 143–146, doi: 10.1307/mmj/1028989917.

[14] S. Kasjan and D. Simson, *Mesh algorithms for Coxeter spectral classification of Cox-regular edge-bipartite graphs with loops, I. Mesh root systems*, in: Fundamenta Informaticae **139** (2015), 153–184, doi: 10.3233/FI-2015-1230.

[15] S. Kasjan and D. Simson, *Mesh algorithms for Coxeter spectral classification of Cox-regular edge-bipartite graphs with loops, II. Application to Coxeter spectral analysis*, in: Fundamenta Informaticae **139** (2015), 185–209, doi: 10.3233/FI-2012-1231.

[16] H. Lenzing and J. A. de la Peña, “Spectral analysis of finite dimensional algebras and singularities”, in: *Trends in representation theory of algebras and related topics*, EMS Ser. Congr. Rep. Eur. Math. Soc., Zürich, 2008, 541–588, doi: 10.4171/062-1/13.

[17] G. Marczak, D. Simson, and K. Zając, “On computing non-negative loop-free edge-bipartite graphs”, in: *Symbolic and Numeric Algorithms for Scientific Computing, SYNASC 2013*, IEEE CPS, Los Alamitos, Calif. [etc.], 2014, 68–75, doi: 10.1109/SYNASC.2013.16.

[18] A. Mróz, *Congruences of edge-bipartite graphs with applications to Grothendieck group recognition II. Coxeter type study*, in: Fundamenta Informaticae **146** (2016), 145–177, doi: 10.3233/FI-2016-1378.

[19] A. Mróz and J. A. de la Peña, *Tubes in derived categories and cyclotomic factors of the Coxeter polynomial of an algebra*, in: J. Algebra **420** (2014), 242–260, doi: 10.1016/j.jalgebra.2014.08.017.

[20] D. Simson, *Mesh algorithms for solving principal Diophantine equations, sand-glass tubes and tori of roots*, in: Fundamenta Informaticae **109** (2011), 425–462, doi: 10.3233/FI-2011-520.

[21] D. Simson, *Mesh geometries of root orbits of integral quadratic forms*, in: J. Pure Appl. Algebra **215** (2011), 13–34, doi: 10.1016/j.jpaa.2010.02.029.

[22] D. Simson, *A Coxeter-Gram classification of simply laced edge-bipartite graphs*, in: SIAM J. Discrete Math. **27** (2013), 827–854, doi: 10.1137/110843721.

[23] D. Simson and K. Zając, *An inflation algorithm and a toroidal mesh algorithm for edge-bipartite graphs*, in: Electronic Notes in Discrete Mathematics **40** (2013), 377–383, doi: 10.1016/j.endm.2013.05.066.

[24] D. Simson and K. Zając, *A framework for Coxeter spectral classification of finite posets and their mesh geometries of roots*, in: Int. J. Math. Math. Sci. **2013** (2013), Article ID 743734, doi: 10.1155/2013/743734.

[25] D. Simson and K. Zając, *Inflation algorithm for loop-free non-negative edge-bipartite graphs of corank at least two*, in: Linear Algebra Appl. **524** (2017), 109–152, doi: 10.1016/j.laa.2017.02.021.

[26] K. Zając, *Numeric algorithms for Corank two edge-bipartite graphs and their mesh geometries of roots*, in: Fundamenta Informaticae **152** (2017), 185–222, doi: 10.3233/FI-2017-1518.

[27] T. Zaslavsky, *Signed graphs*, in: Discrete Appl. Math. **4** (1982), 47–74, doi: 10.1016/0166-218X(82)90033-6.