Comments on the dissertation "Anisotropic least gradient problems" by Wojciech Górny.

April 15, 2020

I will start this referee report by the conclusion. The thesis is sufficient to grant a Ph.D. in Mathematics. Moreover, I really find the thesis is excellent both in merits and in form. Therefore, I strongly recommend to grant it with an honorary distinction. The conclusion is based on the following facts:

First of all, the results presented in the dissertation are very strong. The author has obtained very deep results in the least gradient problem, both in the classical version and in the anisotropic one. In the dissertation he provides new insights to a classical problem in Analysis. For this, he develops some new geometrical techniques, which are, moreover, intuitive and easy to handle with. The author also shows that he masters several tools and fields in Analysis such as Geometric Measure Theory, Calculus of Variations or Optimal Transport as well as some fields in Geometry. All the results have been published or submitted to top journals in the form of 5 different papers in which the author is the sole author in 4 of them. Concerning the results, I find particularly strong the results in Chapter 3 about the complete description of minimizers in the isotropic case, the results in Chapter 4 about existence of minimizers in the 2-dimensional case for a non strictly convex norm and the results in Chapter 7 about existence and regularity of the solutions in the case of a very particular non convex domain.

Concerning the dissertation itself, I find it very well written (apart form the items that I mention below). The Introduction is very nice and comprehensible and the structure of all the Chapters makes it very easy to follow. Even when the proofs are complicated, the author chooses first to show an easier case at the beginning (sometimes the 2-dimensional case) for readability and to get the main idea and then to pass to the general case. The dissertation is full of Examples clarifying the most important results and also showing almost necessity of the hypothesis.

On the other hand, the dissertation could have been written in a more consistent way as a unique document and not just as a recompilation of articles. In particular, there are several repeated things, some misplaced items and some cited results which should have been more detailed within the dissertation.

1. The definition of superlevel sets of solutions E_t is written too many times (even if it's stated that this will be a notation for the whole manuscript).
2. At page 20, the reader is asked to compare Theorem 3.1.1 to some results in the literature. I think it should have been better to state it there in a more detailed way.
3. Some examples are considered twice in different Sections (since they correspond to two different papers). Example 3.4.2. is the same Example 5.3.11 and Example 3.4.3 is the same as Example 5.3.12. I think that only referring to the previous Examples would be better in Section 5.
4. Example 5.3.10 refers to a result in [64]. I think that the author should either provide the construction of the fat Cantor set here or just erase the Example while keeping the paragraph before it.
5. Acknowledgment to the advisor within the core of the dissertation is a very strange thing (it's however, common within the Introduction of a paper). This occurs at Page 113.
6. The results in [65] quoted at the first paragraph os Section 7.2.1 have already been explained before.
7. Page 115. The perimeter of a set has already been introduced.
8. The author should give a proof of Lemma 7.2.7.
9. The author should give the precise definition of the approximating sequence in Page 121.

Concerning the pictures provided, they are very helpful in order to understand some of the proofs. However, I think that more pictures (or more detailed ones) would help in some cases. Otherwise, the reader has to draw a picture by him or herself to be able to follow the proof:

10. Figure 3.1 is confusing since it corresponds to two different results. It should have been better to separate it into two pictures and to add captions to clarify it.
11. In Proposition 3.4.1. a picture both for sufficiency and for necessity of the conditions would help.

12. Figures 4.3, 4.4 and 4.5 could be more detailed (adding either the points or line $m, m', \ldots$.

13. In Proposition 7.4.3 a picture will also help

Finally, I list some minor comments about the dissertation:

13. Page 7. There is no need to recall here the meaning of strictly convex norm.

14. Formulae 1.2.1 and 1.2.2 contain already an inf. So (1.2.1)=(1.2.2) in the sentence after them.

15. Page 11. "the define" is "then define. Immediately after this, please specify that the result is the result of maximization.

16. Theorem 2.1.6. is not clearly written. Does it mean that if $f \in \mathcal{L}^\infty(\partial\Omega)$ and (LGP) admits no solution then, necessarily, f is the characteristic of a fat Cantor set?

17. Lemma 3.2.5. does not state but implies at most countability. In this paragraph, notation for Lebesgue measure should have been fixed before, not at the core of the dissertation.

18. At the beginning of Section 3.2.2, there is an extra "level".

19. Green's formula. It is not clear at all why this is an integration by parts formula without introducing Anzellotti's pairing theory. If the author wants to avoid the notations and results in this theory, then no explanation to the name should be added.

20. Page 32, Step 3a. v is continuous at x, not u.

21. Page 34. The definition of u_0 (as on the left) could be more detailed. For instance stating that u_0 is continuous, that the line segments are the level sets... Moreover, here the author should have stated (as he does later) that he is identifying the hexagon with its vertices.

22. In example 3.4.2, should the condition be $\alpha_2 > \alpha_1$ on (p_3, p_1)?

23. In Corollary 3.5.8, the condition f nonnegative is needed in the hypothesis.

24. The L^1 bound in Proposition 3.5.9 is the same as a previous one and might be skipped.

25. In Corollary 3.5.10. the proof is included in the statement.

26. In Example 4.3.4, please write that p comes from the l_p-norm in the anisotropy. Moreover, please note that the notation l_p is used both as a line segment and as the l_p-norm. Please amend it.

27. Concerning the Definition of uniform convexity at Page 57: does it appear in the literature? Moreover, is Proposition 4.3.9 a new result?

28. At Proposition 4.3.12, E_i is used not supervelel sets but for area minimizing sets.

29. At Theorem 4.3.14. $p'' \in P$ not in ∂P, the same happens with r''.

30. Proof of Lemma 4.4.5. F' is the set bounded by $q_0 q_2 \ldots q_n q_{n+1}$.

31. Proof of Proposition 4.4.8.l', l'' need to be close enough to l for the last sentence in the last but one paragraph to be true (it is not a consequence, but an assumption).

32. Page 70. ACL characterization has never appeared before.

33. Proof of Proposition 4.4.9, note that α should also be fixed (not only ν_0).

34. Proof of Theorem 4.5.1, reference to 4.5.1 is wrong.

35. Proof of Theorem 5.3.1, the reference to Remark 5.3.3. seems to mean that there it is proved that mollification is not defined while this is obvious and the Remark is about when mollification does work.

36. Theorem 5.3.6. and Proposition 5.3.7, part of the proofs are very similar to some above and could be shortened.

37. Before Example 5.4.9. I think that the condition is that Ω is not unbounded in only one direction (and not that it is 1-D).

38. Example 6.2.5, the different use of p could be confusing.

39. Proof of Lemma 7.2.2. Ω is not convex. Ω_- is used here. Moreover, I think that the conclusion comes from the fact that $P(\Omega; \mathbb{R}^2)$ is bounded.

40. Definition 7.3.6. Is ϕ here a Kantorovich potential?

41. Proof of Proposition 7.3.7, please provide a bit more details on the first sentence.

42. Example 7.4.7, can the minimizer be computed?

43. Theorem 7.5.1, some equalities are not definitions but just equalities.

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