UNIVERSITÀ DI PISA

DIPARTIMENTO DI MATEMATICA

Matteo Novaga
Dipartimento di Matematica
Università degli Studi di Pisa
Largo Pontecorvo 5
56127 Pisa
Italy
Tel. +39/050/2213238
Fax +39/050/2210628
e-mail: novaga@dm.unipi.it

May 14, 2020

Report on the PhD Thesis of Wojciech Górny

This thesis deals with anisotropic least gradient problems of the form

$$\min \left\{ \int_\Omega \varphi(x, Du) : \ u \in BV, \ u_{\partial\Omega} = f \right\}$$

where Ω is a (possibly unbounded) strictly convex domain, $\varphi(x, \cdot)$ is a norm depending on x, and f is a boundary datum, not necessarily continuous, which is assumed as trace of the BV-function u.

The main issues considered in this work are existence, uniqueness and regularity of minimizers u, depending on the regularity of φ and f.

The thesis is divided into five main chapters:

In Chapter 3 the author shows that existence might fail if the set Ω is not strictly convex or if the boundary datum f is bounded but not continuous. On the other hand, if Ω is bounded and strictly convex, and the boundary datum lies in BV, then existence holds.

Chapter 4 deals with norms φ in two dimensions, which are independent of x but not necessarily smooth or strictly convex. The boundary datum f is assumed to be continuous. When φ is strictly convex it is shown that the minimizer u is unique and continuous. In the general case there still exists a continuous minimizer, but the uniqueness can be proved only under stronger assumptions on the domain Ω. Without those assumptions there might be other discontinuous solutions, however two minimizers necessarily coincide out of a set where they are both constant.

In Chapter 5, it is shown existence of minimizers if Ω is bounded, φ is strictly convex, and $f \in L^1(\partial\Omega)$ is continuous almost everywhere. It is known that minimizers do no exist in general if f is discontinuous on a set of positive measure.

In Chapter 6 the author considers the case of unbounded boundary data and proves that, if $f \in L^p(\partial\Omega)$ with $1 \leq p < +\infty$, then $u \in L^{\frac{Np}{N-1}}(\Omega)$. This result holds for any bounded open set Ω with Lipschitz boundary.

Finally, in Chapter 7 the analysis is done following a completely different approach, introduced by Rybka, Sabra and the author, and further developed by Dweik and Santambrogio. The basic observation is that, in two dimensions, the (anisotropic) least gradient problem is equivalent to a Beckmann minimal-flow problem, which is in turn related to an optimal transport problem. When Ω is a planar annulus, by studying the corresponding transport problem, the author is able to show existence of minimizers under suitable conditions on the boundary data. Under such conditions, if $f \in W^{1,p}(\partial\Omega)$ with $1 \leq p < +\infty$, then the minimizer u belongs to $W^{1,p}(\Omega)$.

The novelty of this work is the extension of several results which were previously known only in the isotropic case to the general anisotropic setting.

The main tool for attacking the problem is the maximum principle applied to the level-sets of a minimizer u, which are anisotropic minimal surfaces. However, as explained above, in the last chapter the author follows a different approach, based on the reduction of the problem to an optimal transport problem.

My overall impression is that it is an excellent work, both in its merits and form, which provides a clear and complete presentation of the subject, together with new interesting results which led already to several publications.

In conclusion, this thesis surely deserves to be defended, and it is sufficient to grant a PhD. I also recommend to award the PhD title with distinction.

Sincerely,

Matteo Novaga