REPORT ON THE THESIS OF ROBERT ŚMIECH

This thesis consists of six chapters. I'll firstly give highlights of each chapter and then make an evaluation at the end.

1. Chapters 1 and 2

After a brief introduction in Chapter 1, the author first recalled some preliminaries on Mori theory in Chapter 2 and then he presented the famous result of Kebekus-Peternell-Sommese-Wiśniewski (with a supplement by Demailly) on the structure of projective contact manifolds: a projective contact manifold with $b_2 > 1$ must be a (Grothendieck)-projectivised tangent bundle. This section ends with the classification result of Druel: projective toric contact manifolds are either $\mathbb{P}^{2n+1}$ or $\mathbb{P}(T_{\mathbb{P}^1\times\cdots\times\mathbb{P}^1})$.

2. Chapter 3

This chapter is devoted to the study of the following Kawamata-Ambro's effective non-vanishing conjecture:

Conjecture 2.1. Let Z be normal complex variety and Δ an effective $\mathbb{R}$-divisor such that (Z,Δ) is klt. Assume that H is a nef Cartier divisor such that $H-(K_Z+\Delta)$ is big and nef. Then H is effective, namely $H^0(Z,\mathcal{O}(H))\neq 0$.

This chapter deals with the case of Fano manifolds with high index. Namely if Y is a Fano manifold of dimension n with index i_Y and fundamental divisor L. If $i_Y \geq n-2$, then Y is classified (as Mukai varieties, del Pezzo manifolds etc). It follows easily from Hirzebruch -Riemann-Roch formula that $h^0(Y,L)\geq n$. The next case is $i_Y = n-3$, where the author proves that

$$h^0(Y,L)\geq n-2.$$

The proof is divided into several cases: (i) when $n\geq 8$, we can use a result of Wiśniewski to reduce to the case of Picard number one, and then a result of Hwang implies that Y has stable tangent bundle so that we can use a Bogomolov inequality of Langer to get a lower bound via Hirzebruch -Riemann-Roch formula. (ii) when $n\leq 7$, we need furthermore to use a result of Jie Liu on the lower bound of second Chern class and a case-by-case analysis.

The next case of $i_Y = n-4$ is much harder and the author only succeeded in proving the following inequality under the hypothesis $b_2 = 1$

$$h^0(Y,L)\geq n-1.$$

A similar technique is used to get a lower bound of $h^0(Y,-K_Y)$ for singular Y of lower dimension, which reads:

Theorem 2.2. *Let Y be a Gorenstein Fano variety of dimension ≤ 5.*
(i) if Y has only canonical singularities, then $h^0(Y,-K_Y)\geq 4$.
(ii) if Y has only log canonical singularities, then $h^0(Y,-K_Y)\geq 1$.

At the end of this section, the author studies the singularities of a general canonical divisor and he proves:

Theorem 2.3. *Let Y be a Fano variety of dimension 5 with canonical Gorenstein singularities. If a general $D \in |-K_Y|$ is reduced, then D has canonical singularities.*

3. Chapter 4

This chapter is mainly a review of recent progress towards the LeBrun-Salamon conjecture predicting that any Fano contact manifold is rationally homogeneous. The main result is due to Occhetta-Romano-Solá-Conde-Wiśniewski proving this conjecture under the assumption of the existence of a reductive group of higher rank (almost half of the dimension) acting on this Fano contact manifold. After some sktech of the proof, the author relates this approach to the question on the non-vanishing of $H^0(Y, L)$, hence giving a link with the previous section.

4. Chapter 5

This chapter starts with generically contact manifolds (namely contact on an open subset but with a global contact line bundle). An interesting result is that on $\mathbb{P}^{2n+1}$, generically contact structure must be contact. It is a natural question to ask if this holds true for other Fano contact manifolds of Picard number one.

After recalling results on symplectic singularities, the author defines singular contact varieties as an analogue of symplectic varieties. Then it is shown that singular contact varieties correspond to symplectic varieties with $\mathbb{C}^*$-action of weight 1. The author provides some examples of contact varieties by finite quotients. The relation between crepant morphism and contact morphism is revealed. All these properties are parallel to the symplectic case and the proofs are straightforward.

The following nice explicit example is presented: the group $G := \mathbb{Z}_2^n$ acts naturally (by ± 1 on coordinates) on $\mathbb{P}^{2n+1}$ preserving the contact structure, hence $X = \mathbb{P}^{2n+1}/G$ is a contact variety. It is shown that this quotient admits a contact resolution given by $\mathbb{P}(T_{\mathbb{P}^1 \times \cdots \mathbb{P}^1})$.

This section ends with the following result:

Theorem 4.1. *Any projective contact variety of dimension 3 admits a crepant resolution of the form $\mathbb{P}(T_S)$ for some ruled surface S.*

For the proof, we first take a terminalization, which has at most isolated singularities. But the contact condition implies the singular locus is of even codimension, hence it is a crepant resolution. This implies that the resolution is a projectivised tangent bundle of a surface S. Then a detailed analysis shows that S is a ruled surface.

5. Chapter 6

This chapter is devoted to the geometry of Monge-Ampère equations. The author proved two results on the geometry of Lagrangian Grassmanian $\Sigma = LG(3, F)$, where F is a 6-dimensional symplectic vector space and then applied this to the symplectic Monge-Ampère equations (and this application part is out of my expertise).

Let $\Lambda_0^3 F^*$ be the irreducible representation of $\mathrm{Sp}(F)$ corresponds to the 3-rd fundamental weight. By composing with the second Veronese map and the natural projection, we have

$$\mu : \Lambda_0^3 F^* \to \mathrm{Sym}^2(\Lambda_0^3 F^*) \to \mathrm{Sym}^2 F \simeq \mathfrak{sp}(F^*),$$

which gives the projectivisation map

$$\tilde{\mu} : \mathbb{P}(\Lambda_0^3 F) \dashrightarrow \mathbb{P}(\mathfrak{sp}(F)).$$

It is known by Landsberg-Manivel that $\mathrm{Sp}(F)$ acts on $\mathbb{P}(\Lambda_0^3 F)$ with 4 orbits (each with an explicit geometry). The first result of this section gives a description of the fibers and the images under $\tilde{\mu}$ for each of these orbits. The proof consists of some explicit computations. The second result shows that the characteristic variety of a Monge-Ampère equation is in fact defined by the moment map, whose proof relies on the Borel-Weil-Bott theorem computing cohomologies of homogeneous bundles over a rational homogeneous space.

6. Evaluation

Kawamata-Ambro's effective non-vanishing conjecture is one of major conjectures in birational geometry and numerous works are carried out towards it. In this thesis, the author provides new evidence for this famous conjecture in the case of smooth Fano manifolds and its fundamental divisors. This is a new and original result with a clean proof.

The LeBrun-Salamon conjecture on Fano contact manifolds is a deep and difficult conjecture in the intersection of algebraic geometry and differential geometry. Although recent remarkable progress has been made by Buczyński-Occhetta-Romano-Solá-Conde-Wiśniewski, this conjecture is still largely open. This motivates the introduction of contact varieties and then the study of crepant resolution of them (which would provide contact manifolds). I think an in-depth study of these varieties can shed new light to our understanding of Fano contact manifolds.

The last part of the geometry of Monge-Ampère equations is appealing. I'm impressed by the rich geometry associated to the 6-dimensional Lagrangian Grassmanian, whose general hyperplane section is in fact a wonderful completion of $\mathrm{SL}_3/\mathrm{SO}_3$.

In conclusion, this thesis combines lots of technics ranging from minimal model theory, representation theory, contact geometry to PDEs. It's very nice to see that the author successfully conducted high quality researches in the crossroad of these fields. I definitely think this thesis is sufficient to grant a PhD.

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On the revised version:

The author has made numerous improvements to the previous version. Especially the Chapter 4 is completely rewritten, with a new result (Theorem 4.1.3) added. This is much better than the previous version, where it was only a recall of the work of Occhetta-Romano-Solá-Conde-Wiśniewski. Chapter 5 is substantially revised. He has provided a converse of his previous theorem on projective singular contact varieties. To be more precise, he showed that for any ruled surface S, there eixsts a crepant morphism $\mathbb{P}(T_S) \to X$ contracting a section of $\mathbb{P}(T_S) \to S$ to a curve, making X a singular contact threefold. Moreover, he has constructed a threefold with a contact structure but has non-rational singularities. This shows the difference of his definition of contact varieties with that given by Campana-Flenner.

In conclusion, I think this version is much better than the previous one. As said before, I definitely think this thesis is sufficient to grant a PhD.

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