

# Review of the Doctoral Thesis of Piotr Oszer

## *Deformation Theory and Algebraic Group Actions*

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### General assessment

The PhD thesis develops an ambitious and conceptually coherent program in deformation theory and algebraic geometry. Its main idea is to use filtrations induced by one-parameter group actions as an infinitesimal analogue of the Białynicki-Birula decomposition, and then to apply this method to study Hilbert schemes of points, nested Hilbert schemes, Gorenstein algebras, cactus schemes, and Territory schemes. The thesis shows deep technical knowledge in deformation theory and commutative algebra, as well as advanced technical ability to handle state-of-the-art frontier problems.

The main results are contained in Chapter 5, based on the author's single-authored paper. It gives explicit Hilbert-Burch models for Białynicki-Birula cells on  $\text{Hilb}^d(\mathbb{A}^2)$ , proves the conjectural description of Homs and Winz, and obtains formal-local models around arbitrary ideals. These are strong contributions and results. The later chapters contain several interesting new ideas and questions which have not been extensively studied before in the literature: the use of fractal deformation structures, the analysis of unions and connected sums of Artinian algebras, and the introduction of multiplicity-bounded Territory functors.

I very much like Figure ▼, which presents the structure of the dissertation as nine concentric circles descending below the ground level, in the spirit of the *Bellis Hilbertis* in Joachim Jelisiejew's own doctoral thesis. This picture deserves a proper name of its own; I encourage the author to find a good one (*Inferno modulorum*, perhaps?).

### Questions for discussion at the defence

Here I list some questions that I would like to discuss with the candidate during the defence. They are not conditions on my positive recommendation, and I do not request for a revision. Some of them may simply concern conventions or standard deformation-theoretic arguments that I do not fully understand.

1. **The Rees construction in Appendix A.** Let  $(R, F_\bullet)$  be a filtered algebra and let  $\tilde{R} = \bigoplus_{i \in \mathbb{Z}} t^{-i} F_i$  be its Rees algebra. Lemma A.1 states that the Rees construction gives an equivalence between the category of all filtered  $R$ -modules and the category of all graded  $\tilde{R}$ -modules, with inverse given by specialization at  $t = 1$ . Could you clarify the precise category of graded modules here? With the filtration used in Definition 4.48,  $F_i R = \bigoplus_{j \geq i} R_j$  and hence  $t \in \tilde{R}$  since  $1 \in F_{-1} R$ . The nonzero graded module  $\tilde{M} = \tilde{R}/(t)$  then satisfies

$$\tilde{M}/(t-1)\tilde{M} = 0.$$

Thus specialization at  $t = 1$  cannot be an inverse on the category of all graded modules. Do you need a condition on the modules such as  $t$ -torsion-freeness, flatness over the polynomial parameter, saturation, or compatibility with localization?

2. **Negative mixed tangents in Lemma 7.21.** Chapter 7 studies the union  $X \cup_* Y$  of two pointed schemes along their distinguished points. On coordinate rings this is the fibre product

$$\mathcal{O}_{X \cup_* Y} = \mathcal{O}_X \times_k \mathcal{O}_Y,$$

and it carries a bigrading in which the variables coming from  $X$  and  $Y$  have degrees  $(1, 0)$  and  $(0, 1)$ , respectively. Lemma 7.21 states that if  $X$  and  $Y$  are finite  $t$ -level schemes, then  $T_{X \cup_* Y}^1$  has no negative mixed tangent vectors. Here is an example which I would like to understand better. Consider

$$X = Y = \operatorname{Spec} k[z]/(z^3)$$

with the standard grading. Both coordinate rings are generated in degree one and are 2-level, since their socles are generated by  $z^2$ . Their union along the distinguished point has coordinate ring

$$R = k[x, y]/(x^3, xy, y^3).$$

Let  $D = k[\varepsilon]/(\varepsilon^2)$  be the ring of dual numbers and consider

$$R_D = D[x, y]/(x^3, xy, y^3 - \varepsilon x^2).$$

Modulo  $\varepsilon$  this specializes to  $R$ , and the elements  $1, x, x^2, y, y^2$  form a  $D$ -basis, so  $R_D$  appears to be a flat first-order deformation of  $R$ . Equivalently, writing

$$S = k[x, y], \quad J = (x^3, xy, y^3), \quad R = S/J,$$

the deformation is represented by the homomorphism

$$\varphi : J/J^2 \longrightarrow R, \quad \varphi(x^3) = 0, \quad \varphi(xy) = 0, \quad \varphi(y^3) = x^2.$$

It has bidegree  $(2, 0) - (0, 3) = (2, -3)$ , and therefore total degree  $-1$ . It also seems to define a nonzero class in  $T_R^1$ : for a derivation  $\delta : S \rightarrow R$  one has  $\delta(y^3) = 3y^2\delta(y)$  which cannot have a nonzero  $x^2$ -component in  $R$ . Could you explain if this is indeed a negative mixed tangent? If it is excluded, at which point does the construction fail? Is Lemma 7.21 intended to include an additional bound on the degrees of the minimal defining relations, similar to the degree hypothesis imposed later in Setting 7.26?

3. **The hypersurface conclusion in Theorem 7.38.** In Setting 7.26 one takes

$$X = \operatorname{Spec} \operatorname{Apolar}(\Phi), \quad Y = \operatorname{Spec} \operatorname{Apolar}(\Psi),$$

where  $\Phi$  and  $\Psi$  are homogeneous of degree  $t$ . If  $f$  and  $g$  generate the respective socles, then the connected sum is  $X \# Y = \operatorname{Spec}(\mathcal{O}_{X \cup_* Y}/(f - g))$ . In the proof of Theorem 7.38, Diagram (7.39) defines a pullback functor  $P$ , and the proof states that  $P \rightarrow X \times Y$  is an isomorphism on tangent spaces and has one relative obstruction in degree  $-t$ . After proving that this morphism is not smooth, the proof concludes that it is a closed embedding defined by one equation of degree  $-t$ .

I don't fully understand the deformation-theory used in this argument. If complete local rings represent the two functors, why does an isomorphism on tangent spaces imply that the corresponding homomorphism of complete local rings is surjective? Even after showing that the kernel is principal and generated by a homogeneous equation of total degree  $t$ , why must this equation be precisely  $f - g$ ? The degree- $t$  part of  $\mathcal{O}_X \otimes_k \mathcal{O}_Y$  may also contain mixed terms

$$(\mathcal{O}_X)_i \otimes_k (\mathcal{O}_Y)_{t-i}, \quad 0 < i < t.$$

Is there an argument showing that such terms cannot occur?

4. **The conductor and Frobenius-number conventions in Section 8.4.** In Section 8.4 you use  $X = \operatorname{Spec} k[x]$  and  $\mathfrak{m} = (x)$ . Definition 8.33 defines the Frobenius number of a numerical semigroup  $\Gamma$  by  $F_\Gamma = c_\Gamma - 1$ , where  $c_\Gamma$  is the conductor. Definition 8.34, however, calls  $\operatorname{rk}_{\mathcal{O}_S} \left( \frac{\mathcal{O}_X \otimes_k \mathcal{O}_S}{\mathfrak{m}^{c_Y}} \right)$  the Frobenius number of a Territory point. At a  $k$ -point with associated semigroup  $\Gamma$ , this quantity seems to be  $\dim_k \frac{k[x]}{(x^{c_\Gamma})} = c_\Gamma$ , rather than  $F_\Gamma = c_\Gamma - 1$ . Is Definition 8.34 intended to define the conductor number rather than the Frobenius number? Moreover, let

$$\mathcal{C}_\Gamma = x^{c_\Gamma} k[x] \subset k[\Gamma]$$

denote the conductor ideal. Since every integer at least  $c_\Gamma$  belongs to  $\Gamma$ , one has

$$\dim_k \frac{k[x]}{\mathcal{C}_\Gamma} = c_\Gamma.$$

If  $\delta$  is the genus of  $\Gamma$ , then exactly  $\delta$  integers in  $\{0, \dots, c_\Gamma - 1\}$  are gaps, and hence

$$\dim_k \frac{k[\Gamma]}{\mathcal{C}_\Gamma} = \#\{\gamma \in \Gamma : \gamma < c_\Gamma\} = c_\Gamma - \delta.$$

Shouldn't  $F_\Gamma$  and  $F_\Gamma - \delta$  in the proof of Theorem 8.42 and in formula (8.43) be replaced by  $c_\Gamma$  and  $c_\Gamma - \delta$ ?

## Conclusion

The dissertation contains strong original and important results. The published paper and preprint on Hilbert-Burch matrices, the non-reducedness of the Gorenstein locus, and the general deformation-theoretic viewpoint represent important contributions to the field. The candidate demonstrates technical power and independence well beyond what is expected of a doctoral researcher. In my opinion, the dissertation satisfies the requirements for the doctoral degree in mathematics, and I give my unconditional recommendation for Piotr Oszer to be admitted to the public defence.



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