

Reviewer's report on Piotr Oszer's doctoral thesis

Overall evaluation

The overall evaluation of the thesis is positive. The candidate clearly demonstrates a high level of scientific maturity and a thorough understanding of the field to which his research topics belong.

The development of the thesis (effectively demonstrated in the figure on page 2) also clearly illustrates the candidate's growth.

- The initial part (Chapter 5), which pertains to the Hilbert scheme of points in the plane, contains relatively little original content. Nevertheless, it has enabled the candidate to explore and assimilate a plethora of highly beneficial concepts and techniques from the domains of algebraic geometry, commutative algebra, and representation theory.
- Chapters 6 and 7 present original concepts that yield highly intriguing outcomes in a research domain that is currently experiencing significant activity and interest, but also substantial competition. The candidate's results contribute to the advancement of the discipline and are of interest to the relevant community.
- Chapters 8 and 9 could mark the starting point for an independent line of research. In fact, the methods and tools developed in the first part are applied to a less well-known and less studied moduli space. While the results presented here are consistent with those previously obtained, they are also very promising and could lead to significant future developments.

I therefore conclude that the candidate has met the scientific requirements for the award of the Ph.D. degree.

Request for review, specific comments, suggestions

Two aspects require careful review.

1. A significant number of works and results concerning the study of the Hilbert scheme of points in the plane and the relationship between Gröbner cells and Białynicki-Birula cells have been overlooked. Some of these address questions that were presented as unsolved in the dissertation.
2. The quality of the thesis writing could be improved to meet the standards expected for this kind of document.

In the following section, I outline the specific points that should be addressed.

Hilbert-Burch matrices/Gröbner cells/Białynicki-Birula cells

References

Here is a list of some articles that address topics closely related to the thesis but were not taken into consideration.

[NS] Notari, Spreafico. *A stratification of Hilbert schemes by initial ideals and applications*. Manuscripta Math. 101 (2000), no. 4, 429–448.

- [FR] Ferrarese, Roggero. *Homogeneous varieties for Hilbert schemes*. International Journal of Algebra, Vol. 3, 2009, no. 9-12, 547–557.
- [RT] Roggero, Terracini. *Ideals with an assigned initial ideals*. Int. Math. Forum, Vol. 5, 2010, no. 53-56, 2731–2750.
- [LR] Lella, Roggero. *Rational components of Hilbert schemes*. Rend. Semin. Mat. Univ. Padova 126 (2011), 11–45.
- [Led] Lederer. *Gröbner strata in the Hilbert scheme of points*. J. Commut. Algebra 3 (2011), no. 3, 349–404.
- [LR2] Lella, Roggero. *On the functoriality of marked families*. J. Commut. Algebra 8 (2016), no. 3, 367–410.
- [KL] Kambe, Lella. *The Gröbner fan of the Hilbert scheme*. Ann. Mat. Pura Appl. (4) 200 (2021), no. 2, 547–594.
- [Kam] Kambe. *Computable Bialynicki-Birula decomposition of the Hilbert scheme*, Saitama Mathematical Journal, Vol. 34 (2022), 1–17.
- [BCL] Bertone, Cioffi, Lella. *Free resolutions and marked families*. Journal of Commutative Algebra (2026).

Comments

The first point of confusion arises in Chapter 3, page 9. A dictionary is, by definition, bidirectional, but this is not the case for $\mathbb{Z}$ -gradings and monomial orders. A $\mathbb{Z}$ -grading induces only a partial order on monomials, whereas a monomial order is a total order. In particular, for every monomial order $\geq$ there exists a $\mathbb{Z}$ -grading such that $\lim_{t \rightarrow 0} t \cdot [I]$ is the (monomial) initial ideal $J = \text{in}_{\geq}(I)$. However, in general $\lim_{t \rightarrow 0} t \cdot [I]$ need not be a monomial ideal. To obtain such a dictionary, some additional assumptions are needed, and these are not sufficiently explained.

Secondly, it should be noted that a significant proportion of the methods developed in Chapter 5 can be handled more efficiently with the theory of marked families and Gröbner strata developed over the past 15 years (see references above). While I have full confidence in the candidate's ability to achieve such results independently, it is worth noting that this general theory was not discussed. It should be noted that some constructions are not fully justified in the thesis. For example, Setting 5.5, Definitions 5.13 and 5.19 appear to be somewhat arbitrary and inaccurate. Conversely, the theory of marked bases provides a rationale for their validity, particularly in the context of studying deformations of the ideal.

Let us consider the example (x^2, y^3) . The candidate does not explain why he considers a 3×2 syzygy matrix and a set of generators that is not minimal. In fact, the candidate uses the Pommaret basis with the implicit convention that $x > y$.

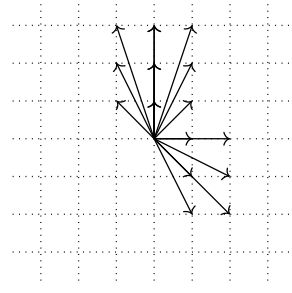
$$\{x^2, xy^3, y^3\} \quad \begin{array}{c} \bullet \\ \bullet \\ \circ \\ \circ \\ \circ \\ \circ \end{array} \quad \begin{bmatrix} -y^3 & 0 \\ x & -1 \\ 0 & x \end{bmatrix}$$

Analyzing the deformations of monomials provides additional information about the syzygy matrix. In order to deform the ideal, one can consider the *marked set*

$$\begin{aligned} f_{x^2} &= x^2 + a_{(2,0)}^{(1,2)}xy^2 + a_{(2,0)}^{(0,2)}y^2 + a_{(2,0)}^{(1,1)}xy + a_{(2,0)}^{(0,1)}y + a_{(2,0)}^{(1,0)}x + a_{(2,0)}^{(0,0)} \\ f_{xy^3} &= xy^3 + a_{(1,3)}^{(1,2)}xy^2 + a_{(1,3)}^{(0,2)}y^2 + a_{(1,3)}^{(1,1)}xy + a_{(1,3)}^{(0,1)}y + a_{(1,3)}^{(1,0)}x + a_{(1,3)}^{(0,0)} \\ f_{y^3} &= y^3 + a_{(0,3)}^{(1,2)}xy^2 + a_{(0,3)}^{(0,2)}y^2 + a_{(0,3)}^{(1,1)}xy + a_{(0,3)}^{(0,1)}y + a_{(0,3)}^{(1,0)}x + a_{(0,3)}^{(0,0)} \end{aligned}$$

The bi-grading (5.6) can be extended to the marked set in order to preserve homogeneity as follows

$$\deg \left(a_{(i,j)}^{(l,k)} \right) = (i,j) - (l,k).$$



Then, the theory of marked families tells how to lift the two syzygies among the monomials to two syzygies among the marked set. It turns out that the syzygy matrix (the associated Hilbert-Burch matrix) should have the following shape

$$\begin{bmatrix} -y^3 + c_{1,1,2}y^2 + c_{1,1,1}y + c_{1,1,0} & c_{1,2,2}y^2 + c_{1,2,1}y + c_{1,2,0} \\ x + c_{2,1,2}y^2 + c_{2,1,1}y + c_{2,1,0} & -1 + c_{2,2,1}y + c_{2,2,0} \\ c_{3,1,2}y^2 + c_{3,1,1}y + c_{3,1,0} & x + c_{3,2,1}y + c_{3,2,0} \end{bmatrix}$$

where the coefficients $c_{i,j,k}$ are rational functions of the coefficients describing the marked set. The bi-grading of coefficient $c_{i,j,k}$ can be deduced by the bi-graded structure of the resolution of the monomial ideal (Proposition 5.10)

$$0 \rightarrow \mathbb{S}(-2, -3) \oplus \mathbb{S}(-1, -3) \rightarrow \mathbb{S}(-2, 0) \oplus \mathbb{S}(-1, -3) \oplus \mathbb{S}(0, -3) \rightarrow E \rightarrow 0.$$

that gives the following bi-grading on the entries of the syzygy matrix

$$\begin{bmatrix} (2,3) - (2,0) & (1,3) - (2,0) \\ (2,3) - (1,3) & (1,3) - (1,3) \\ (2,3) - (0,3) & (1,3) - (0,3) \end{bmatrix}.$$

This example highlights a problem in the construction presented in the thesis. In fact, it appears that there are more admissible indices than those obtained from Definition 5.13 (which are necessary to study all deformations of the ideal). It is essential that this issue is addressed by the candidate. This is very important because, by correctly choosing the admissible indices, the conditions on the parameters $c_{i,j,k}$ that give rise to a flat deformation of the monomial ideal are obtained by requiring

$$\begin{bmatrix} f_{x^2} & f_{xy^3} & f_{y^3} \end{bmatrix} \begin{bmatrix} -y^3 + c_{1,1,2}y^2 + c_{1,1,1}y + c_{1,1,0} & c_{1,2,2}y^2 + c_{1,2,1}y + c_{1,2,0} \\ x + c_{2,1,2}y^2 + c_{2,1,1}y + c_{2,1,0} & -1 + c_{2,2,1}y + c_{2,2,0} \\ c_{3,1,2}y^2 + c_{3,1,1}y + c_{3,1,0} & x + c_{3,2,1}y + c_{3,2,0} \end{bmatrix} = 0$$

and eliminating the coefficients a 's. In this example, we obtain

$$\begin{aligned} c_{2,2,1} &= -c_{2,1,2}c_{1,2,2}, & c_{2,2,0} &= -c_{2,1,1}c_{1,2,2} - c_{2,1,2}c_{1,2,1} - c_{1,1,2}c_{2,1,2}c_{1,2,2}, \\ c_{3,2,1} &= -c_{3,1,2}c_{1,2,2}, & c_{3,2,0} &= -c_{3,1,1}c_{1,2,2} - c_{3,1,2}c_{1,2,1} - c_{1,1,2}c_{3,1,2}c_{1,2,2}. \end{aligned}$$

i.e. a family of Hilbert-Burch matrices parametrized by the affine space $\mathbb{A}^{12}$ describing an open subset of $\text{Hilb}_6(\mathbb{A}^2)$. In particular, this computation shows that some of the problems that lead to “the failure of the dream scenario” are due to an inaccurate definition of the spread out matrix (Definition 5.19). Indeed, the matrix considered in Example 5.30 with the right definition would not arise. The condition $c_{2,2,1} = -c_{2,1,2}c_{1,2,2}$ forces us to consider the matrix

$$\begin{bmatrix} -y^3 & c_{1,2,2}y^2 \\ x + c_{2,1,2}y^2 & -1 - c_{2,1,2}c_{1,2,2}y \\ 0 & x \end{bmatrix}$$

whose minors are

$$f_{x^2} = x^2 + c_{2,1,2}xy^2, \quad f_{xy^3} = xy^3, \quad f_{y^3} = y^3 - c_{1,2,2}xy^2.$$

As pointed out in the thesis, this 12-dimensional family does not correspond to a Białynicki-Birula cell because the arrows in the picture at the top of page 3 are not contained in any half-plane. However, the answer to Question 5.48 can be deduced from the article [LR2], which explains in which cases the marked family can be realized as a Gröbner stratum.

Other mathematical issues

- The thesis introduced certain terms which are not entirely effective because they do not help the reader build an intuitive understanding.
 - The use of the term *pointed Hilbert scheme* is somewhat misleading, as a pointed scheme is a scheme with a point, whereas a pointed Hilbert scheme is not a Hilbert scheme with a point. Rather, it is a nested Hilbert scheme parametrizing pointed schemes. In addition, it bears a strong resemblance to a *punctual Hilbert scheme*, which denotes a different object.
 - It appears that the use of the word *fractal* may not be entirely accurate, given that the self-similarity phenomenon under consideration does not repeat itself at scales which are arbitrarily small.
- The thesis is lacking in meaningful examples. A more systematic use of the $\mathbb{G}_m$ -action to study deformations is promising, but the thesis would benefit from including more examples that clearly illustrate when the technical tools developed provide better control and understanding than the classical approach.
- The *Macaulay2* code contained in the appendix is not functioning as intended. This is due to the fact that some of the functions it relies on have not been defined. Nowadays, this approach is not aligned with best practices for sharing code. The creation of an online repository would be a more efficient solution.

Clarity of presentation

The following changes have been proposed to enhance the manuscript, with the most straightforward modifications listed first.

- There are numerous typos and ineffective phrases in English that can be successfully corrected with the (supervised) use of artificial intelligence.
- The use of LaTeX can be significantly improved. In particular, readability would improve greatly with proper use of the `\theoremstyle{}` command: `theorem` for theorems/ propositions/..., `definition` for definitions/notation/examples/..., and `remark` for remarks/comments/....

The numbering style in the introduction and in the main text is the same.

- The introductions, both the general one and those for the individual chapters, are not very effective; in fact, they often cause a great deal of confusion. Introductions should help the reader to place the subsequent content within the broader context and identify the issues that are addressed. In this thesis, however, the introductions are often brief synopses of the subsequent content.

There are two major issues.

1. The repeated use of duplicate numbering for the same item (proposition/theorem/definition) is to be avoided. In the event that a mathematical item is mentioned in an introduction with content identical to that in the main text, it must have the same numbering. In certain cases, it may be advisable to employ a different numbering system if a theorem or definition is mentioned in the introduction in a simplified form for the purpose of the introduction.

2. The repeated use of mathematical terminology that has not yet been defined. For instance, on page 30, in Example 5.47 and in Question 5.48, reference is made to Definition 5.8, which appears on the following page (but this same issue occurs very frequently throughout the thesis).
- The main introduction should be structured around an appropriate contextualisation of the work, rather than starting with notations and definitions.
 - At present, Part I (Methods) of the thesis is not clear enough. When writing a doctoral dissertation, there are two possible approaches:
 1. include a section on prerequisites that should compile concepts and results from multiple sources and synthesise them for the purposes of the dissertation. It should adequately explain the importance of the prerequisites and provide a sufficient number of examples;
 2. provide references to the sources where the prerequisites can be found, without listing them explicitly (this is often used when the dissertation is a collection of previously published articles).

This thesis seems to fall under the first case, but the extensive list of definitions and theorems is not well connected, which makes it difficult to comprehend the overall meaning. Furthermore, some methods are referenced only in one part of the thesis, many pages away (e.g. apolarity is introduced at pag. 2 and is only used in Chapters 6 and 7). It would be much more effective to introduce and explain a method shortly before it is used.

Line-by-Line comments

This is a list of minor corrections that I have noted during my reading of the thesis (certainly incomplete).

pag. 1, second-to-last line. Add some details about the conjecture.

pag. 2, figure. It's an effective idea, but the latex version could be improved.

pag. 4 The page displays both $\text{Hilb}_{\bullet, \bullet+1}(Y)$ and $\text{Hilb}_{\bullet-1, \bullet}(Y)$. It's a little confusing.

pag. 1, chapter 2, first paragraph english

pag. 1, chapter 2 Hedgehog point is not defined

pag. 5, Definition 3.11 It is a little strange to talk about elementary components without having introduced the smooth-able component

pag. 5, Definition/Notation 3.26 centred $\rightarrow$ centered

pag. 13, after Definition 4.16 english

pag. 21, Lemma 4.58 $T^2(\mathcal{O}_X, \mathbb{k})_{<-m} = 0$

pag. 28, first line author's article

pag. 31, eq. (5.11) $\xrightarrow{M_E} \rightarrow$

pag. 43, Conjecture 1 some reference is missing

pag. 48, Proposition 6.6 There is

pag. 53, paragraph after Definition 6.15 $\text{Hilb}_{1,\delta}(\mathbb{A}^n)$

pag. 53, Definition 6.17 ... over **an** irreducible scheme ...

pag. 53, Definition 6.31 is there any assumption on n ?

pag. 60, first line of the introduction ... the Hilbert scheme $\text{Hilb}_\delta(\mathbb{A}^n)$...

pag. 61, after the first formula Y should be U

pag. 80, Definition 7.45 the secant variety has been already defined

pag. 84, after eq. (8.1) the integral closure has not been defined

pag. 84, beginning of section 8.2 filed $\rightarrow$ field

pag. 84, Definition 8.2 ... the module $\mathbb{A}$... $\rightarrow$... the module $\mathcal{A}$...

pag. 89, Theorem 8.20 it seems that $\text{Ter}_X^{\delta,r}$ is defined in the statement.

pag. 96, diagram It doesn't make much sense, since the equivalences are all proven later

pag. 96, Definition 8.46 hard to understand without at least some context

pag. 99, beginning of section 9.1 ... non-reduced. Even if ...

pag. 125, Sza23 poiants $\rightarrow$ points

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Paolo Lella PhD
Associate Professor
Politecnico di Milano
Dipartimento di Matematica
Piazza Leonardo da Vinci 32, 20133 Milano

Firmato digitalmente
da: PAOLO LELLA
Organizzazione:
POLITECNICO DI
MILANO/80057930150