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Report on the PhD Thesis

Singularities of minimizing harmonic maps into closed manifolds

by Michal Miskiewicz

In his Phd thesis Michal Miskiewicz studies minimizing harmonic maps into closed manifolds. These maps play an important role in the field of geometric analysis since they are used as a toy model when studying regularity issues for systems of partial differential equations satisfying the constraint that the solutions have to map into a given manifold. There is a vast literature on this topic starting in the 60's with the work of Morrey and then there was a huge boost in the 80's via the works of Schoen and Uhlenbeck. They obtained very nice results about the regularity and about the behaviour near possible singularities of minimizing harmonic maps. These results were later extended by Simon, Lin and Hardt (just to mention a few names) and in another direction by Helein, Evans and Bethuel to so called stationary harmonic maps. The last few years substantial advances in the study of singularities have been obtained by Naber and Valtorta and their results and ideas play a prominent role in the thesis under consideration.
The thesis of Mr. Miskiewicz is mainly concerned with a detailed study of the behavior near a singularity of these minimizing harmonic maps. They only arise when the domain manifold has dimension at least three and hence this is the setting of this thesis. In the first two chapters of the thesis Mr. Miskiewicz gives a very nice introduction into the field of harmonic maps and he surveys all the important and necessary results for the following chapters. I really want to emphasize that I have read plenty of overview articles and books about harmonic maps and related topics and I have rarely seen such a nice presentation of this topic. This is very remarkable for the introduction of a PhD thesis.

The content of chapter three of the thesis is a paper of the author which was published an the arXiv precisely one year ago. There the author extends earlier works of Hardt-Lin and Lin-Wang in which it was shown that the singular set of an energy minimizing harmonic map is a union of a finite set and a set of Hölder continuous objects, depending on the dimension. This result is shown for maps from the unit ball in $\mathbb{R}^4$ into S^2 resp. from the unit ball in $\mathbb{R}^5$ into S^3. Mr. Miskiewicz was able to extend this for maps from the unit ball in $\mathbb{R}^n$ into a general closed Riemannian manifold but of course he had to pay a prize for this generality. Namely, he was only able to characterize the part of the singular set for which the energy density satisfies a certain upper bound. And this assumption was used crucially in the proof. On the other hand, it is very interesting to see that he was indeed able to show this result in the general setting.

Chapter 4 deals with the above mentioned important work of Naber and Valtorta on the structure of the singular set of harmonic maps. Again, the author summarizes the key results and ideas of the papers of Naber and Valtorta in a very understandable way. These results also play a key role in the remaining three chapters of the thesis.

First, in chapter 5, the author extends the discrete Reifenberg theorem of Naber and Valtorta in a crucial way. These results have alreay been published in the journal *Ann. Acad. Sci. Fenn. Math.*

Finally, in chapters 6 and 7, the author presents the results of his two papers with K. Mazowiecka and A. Schikorra which were posted on the arXiv in November 2018 resp. February 2019. In these papers the authors extend a famous result of Almgren and Lieb. They showed that the singular set of a minimizing harmonic map u from $\Omega \subset \mathbb{R}^3$ into S^2 with boundary data $u|_{\partial\Omega} = \varphi \in W^{1,2}(\partial\Omega, S^2)$ satisfies the bound

$$\#\mathrm{sing}(u) \leq C(\Omega) \int_{\partial\Omega} |\nabla\varphi|^2.$$

Here, the result is extended to maps from an arbitrary domain $\Omega \subset \mathbb{R}^n$ into S^2. Of course the left hand side of the estimate then has to be replaced by the $(n-3)$-dimensional Hausdorff measure of the singular set. The proof of the result of Almgren and Lieb relied heavily on the fact that the so called tangent maps have all been classified by Brezis, Coron and Lieb in their setting. This is not true anymore once one considers higher dimensional domains and here the proof relies on the estimates of Naber and Valtorta which were presented in chapter 4.

Finally, in the last chapter a stability result for singularities is shown. More precisely, given a minimizing harmonic map $u : \Omega \to S^2$ with boundary values φ and sequences of minimizers u_k with corresponding boundary values φ_k which converge to u resp. φ in an appropriate sense, it is shown that the singular sets converge with respect to the Wasserstein distance. This implies in particular that the corresponding Hausdorff dimensions of the singular sets converge. The proof of this result relies again on the work of Naber and Valtorta.

In summary, the thesis is extremely well written and it contains plenty of new and interesting results. It is very fascinating to see how fast the author was able to understand and apply the fundamentally new ideas of Naber and Valtorta to open problems in this active field of research.

Thus, I strongly recommend to accept the thesis in its current form and I would definitely recommend to grant ”a honorary distinction”. I really find this thesis outstanding.

Best Regards,

Prof. Dr. Tobias Lamm