

Antonin Chambolle  
CMAP, Ecole Polytechnique and CNRS,  
91128 Palaiseau  
France  
antonin.chambolle@cmap.polytechnique.fr

26th June, 2019

Report on the Doctoral Dissertation of Michał Łasica, entitled  
“Parabolic equations with very singular nonlinear diffusion”,  
written under the supervision of Profs. Mucha (Warsaw) and Moll (Valencia),  
University of Warsaw, Poland.

This Doctoral Dissertation gathers four important studies on a class of singular/degenerate parabolic equations with (very) fast diffusion, related to the gradient flow of the total variation functional. It is very well written and pleasant to read. A first part introduces the main results, with detailed explanations, the technical results and proofs are then in four separate sections numbered 2 to 5.

The first part is about a construction of the  $TV_1$  flow with “rectilinear” polygons in 2D. The  $TV_1$  is the anisotropic total variation based on the 1-norm  $\int |D_x u| + |D_y u|$ . The corresponding “Wulff shape” is the  $L^\infty$  ball and solutions are expected to develop vertical and horizontal slopes or edges. This is carefully analysed (as well as a related sort of “Cheeger” problem which allows to characterize the evolution of the facets). The geometry of this case is interesting, as Michał Łasica is able to show that if the initial datum is a characteristic of a rectilinear shape (with only horizontal and vertical edges) then the solution of the related variational problem, and the flow, keeps the same structure with edges aligned along the same horizontal and vertical lines (yet which may grow). In particular, facets remain faceted yet may break into several pieces along existing edges. In this context, Theorem 1.2 is particularly interesting as it shows that despite the fact discontinuities may appear, still some kind of continuity is preserved (the maximal size of a discontinuity is nonincreasing, and if the initial data is continuous it remains so).

The next problem which is addressed is the study of a nondegenerate parabolic equation which mixes the heat equation and the total variation flow (in 1D). In this case still, due to the singularity at  $u_x = 0$  of the total variation, fast diffusion can happen and facets are created. Thm 1.3 gives a precise description of these facets.

The third problem studied by Michał Łasica is a *local* estimate for a 1D total variation flow of vectorial functions. This is not much studied, although quite interesting. In particular, Michał Łasica’s main result on this problem is a bit unexpected and very interesting. It shows that the size of the derivative  $|\mathbf{u}_x|$  is nonincreasing (pointwise, or more precisely as a measure). This is quite obvious for scalar valued functions, but I find it a bit unexpected in the vectorial case which is a bit more complex. The proof



is obtained through a careful analysis of a regularized version of the flow and a priori estimates, it is quite short and elegant.

The last part of Michał Lasica's work is about the TV flow of functions with values in a manifold. This is a very interesting (and difficult) problem. Michał Lasica studies regular solutions (Lipschitz), which is reasonable as jumps are even less understood (and can be defined in various ways depending on whether one uses an intrinsic total variation or the one relative to an embedding). This part contains several very interesting results, where the manifold's curvature plays an important role. In particular, the flow for positively curved manifolds may become singular. The first results are existence and uniqueness result (up to a possible time where regularity is lost). The existence is shown by first studying a regularized flow with a parameter  $\varepsilon > 0$  (as usual  $|\nabla \mathbf{u}|$  is replaced with  $\sqrt{\varepsilon^2 + |\nabla \mathbf{u}|^2}$ ). There are some small issues to prove the existence result for  $\varepsilon > 0$  with arbitrary initial data and the regularity discussed in the thesis (see point 5 in the comments below). The result should be true for nice enough data initial (for instance, compactly supported), or with less smoothness. This is not so relevant, as in the limit  $\varepsilon \rightarrow 0$  one cannot hope better than Lipschitz functions  $\mathbf{u}(t)$ . Next, Michał Lasica shows that solutions close to a point will stay close for some time, and converge to a constant if close enough, and as expected in finite time. This is far less obvious in the manifold-valued case than in the standard case; I find this result very interesting (also the dependence on the sign of the curvature of the target manifold, which perfectly makes sense).

If the sectional curvature is negative, the flow behaves better, in particular Theorem 1.8 shows that it goes to a constant, while Theorem 1.9 studies the asymptotic of a flow from a manifold into a manifold. If the target manifold has negative curvature, again the flow exists for all time, and in particular converges to a 1-harmonic map which might be nontrivial. This is a very nice result.

To sum up, this works contains a series of original results which are non standard and difficult, as the flows which are studied have very weak regularizing properties and the evolutions are sometimes nonlocal. It mixes in a nice way geometry and analysis, and especially the last part requires a delicate understanding of the manifolds' properties. I think that clearly this work, which contains high level geometric analysis and a few impressive results, can be defended. I also think that this PhD could be granted with an honorary distinction, considering that it contains already a few interesting papers published in very good journals.

Antonin Chambolle



**Typos and comments:**

1. P. 12, line 7 from bottom, (1.21)  $\rightarrow$  (1.22);
2. P. 13, is (1.27) really the correct setting, with *both*  $a^\bullet$  and  $b^\bullet$  ordered?
3. Lemma 2.2: a a few typos in the proof of this lemma, where  $F$  often stands for  $F_0$ , and  $L$  for  $L_0$  (at pages 31, 32, 33). In (2.18)  $\liminf$  could be replaced with  $\lim$ .
4. On page 46 it would be nice to recall the definition of  $\mathcal{J}$ .
5. Section 5.3, Step. 1:  $\mathbf{u}_0$  needs to depend on  $\varepsilon$ , because of (5.20). A construction of a compatible  $\mathbf{u}_0$  for  $\varepsilon > 0$  is supposed to be shown in Lemma A.2, however the function built in that lemma does not satisfy (5.20) for positive  $\varepsilon$ . It means there is should still be some work to do to build a compatible initial data  $\mathbf{u}_0$  and write a full proof of Thm. 1.6. Lemma A.2 should be true if the initial data has some smoothness, but is probably not easy to show.
6. p. 66, 14th line, should the “+” sign not be rather “=”?

