Report

on a doctoral dissertation written by Michał Brzozowski

"Sharp weighted inequalities for martingales"

The presented doctoral dissertation is devoted to obtaining estimates either for martingale transforms, or, more general, for differentially subordinated martingals in various weighted L^p-spaces. But before to describe the results obtained in this dissertation I would like to say a few words about historical context.

I start with the discovery of the criterion of boundedness for some classical singular integral operators in weighted L^p-spaces. It is so called Muckenhoupt A_p condition. Tremendous development of the theory of singular integral operator starts from this point. It appears that the same A_p condition guarantees boundedness, roughly speaking, for all Calderòn–Zygmund operators. Various types of space decomposition were the first tool of the proof. This tool by its nature usually cannot give us the sharp constants in the inequalities under investigation. The method allowing to obtain the sharp constants is so-called Bellman function method. It originated in stochastic control theory, but at the first time appears in 80's in works of Burkholder without mentioning of the control theory and even the Bellman equation. But in fact Burkholder solved the Bellman equation related to the problem under consideration. Solution of the Bellman equation supplies us with the function that was later called the Bellman function of the corresponding extremal problem. If you would like to find supremum of some quantity under some fixed parameters, then the maximal value of this quantity is just by definition the value of the Bellman function for the corresponding extremal problem. Your fixed parameters are argument of this function. When the extremal problem is self-similar under scaling, the definition of the Bellman function immediately leads to some inequality of concavity type, and the Bellman function appears to be minimal possible among all the functions satisfying this inequality. This extremal property implies that the concavity must be degenerated in some direction and in such a way we come to a nonlinear partial differential equation for our Bellman function. This equation sometimes can be solved, especially if your Belmann function is a function of two or three variable. I know the only case when this equation was solved in the case of four variables in a convex domain. For extremal problems in weighted L^p-spaces the corresponding Bellman functions are usually functions of four and more variables. Moreover, the domain is not convex, what causes additional significant difficulties.

When to find the Bellman function is too difficult then it is possible to try to find a majorant satisfying the same concavity and obstacle conditions, but not obligatory satisfying the Bellman equation. In analysis such a function is usually called a supersolution and in probability it is usually called an auxilliary function (after Burkholder, who used this term for Bellman function as well). However, in the literature sometimes supersolutions are also called Bellman functions, but in fact the Bellman function is minimal possible supersolution and it is unique for a given extremal problem. Such a supersolution can replace the true Bellman function in the standard machinery of obtaining an estimate. The only problem is in the fact that the Bellman function supplies us with a sharp estimate by the definition, but the constant we get using a supersolution is not obligatory sharp. Roughly speaking a supersolution is sufficient to prove boundedness of an operator, but it is in general not sufficient to get the value of its norm. If we are lucky, our supersolution can give us some asymptotic sharpness with respect to some parameter. This is just the case of the presented work: the author was succeeded in constructing such supersolutions in four different problems that supplies us with the sharp dependence the norm of the corresponding operator with respect to A_p-characteristic.

Let us turn now to the structure of the dissertation. After the first introductory chapter, the second chapter contains the proof of the $L^p(w)$-estimate of the martingale transform for A_p-weights w with the sharp dependence of the norm with respect to the A_p-characteristic of the weight. For this aim a supersolution of four variables is constructed. The result itself is not new, this is another proof of the result of Domelevo and Petermichl. However, in the dissertation there is a rather important progress in the problem. The author was succeeded in presenting an explicit analytic expression for a supersolution supplying us with the mentioned sharp dependence of the norm with respect the A_p-characteristic of the weight. The supersolution from the paper by Domelevo and Petermichl is not explicit, in fact, by using some approximation procedure they proved only existence of an appropriate supersolution.

In the following three chapters not only the found supersolutions are new, but the resulting estimates are new as well. The main result of Chapter 3 is constructing a supersolution to obtain weak (p,p)-estimate for martingale transform in the weighted case for A_p-weights. Since the author is interested only in sharp dependence of the norm from the A_p-characteristic but not in the numerical constant, he restricts himself with the case $1 < p < 2$, because for $p \geq 2$ the desired inequality follows from the strong-type estimate. In result he covers all exponents $1 < p < \infty$. The end-point cases $p = \infty$ and $p = 1$ as usual have their own specific and they are considered separately in the last two chapters of the dissertation. In Chapter 4 weak (∞,∞)-estimate is considered for subordinated martingale, i. e., automatically it is true for martingale transforms. For this aim in this chapter a supersolution is constructed with a bit stronger concavity property then in the preceding chapters. In Chapter 5 the opposite end-point case $p = 1$ is considered, but in strong setting. So, it can be considered as a limit case of the inequality from Chapter 2. But martingale transform itself is unbounded for $p = 1$, and the correct analog is the inequality for maximal operator applied to the martingale transform. Here the supersolution of five variables was constructed, an additional variable appears due to the maximal operator involved.

Now I would like to say about mathematical qualification of the author. I see that the author demonstrate high level qualification from various points of view. First, about representation of material. The text is well structured and written very clear, this is especially important because in the dissertation there is a lot of technicalities. This is the first case in my long mathematical practice when after reading such a long text I marked no misprint. However it is more important to say about mathematical skill demonstrated by the author. Those who are not familiar with the Bellman function technique could see it the dissertation a huge amount of rather elementary calculation and decide that this is the main work made by the author. But in fact the main work is hidden. The main work is to find the necessary Bellman function or supersolution. The verification of the required properties is a more or less clear routine. It can be rather difficult, sophisticated and cumbersome, but real difficulties are in finding the needed function. It is a piece of art, and it requires a good intuition. The intuition does not appears itself. The fact that the author has constructed four rather nontrivial supersolutions means that dozens of other Bellman functions, maybe more simple, were studied in details by him to obtain the necessary intuition.

In conclusion I would like to say that by my opinion the author of presented dissertation deserves the Ph.D. in mathematics. Taking into account the mathematical significance of the thesis and the clarity of the presentation, I would like to propose a distinction for the dissertation.

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