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Referee report

on the doctoral thesis entitled
“Sharp weighted inequalities for martingales”
written by Mr. Michał Brzozowski
under the supervision of Accos. Prof. Adam Osękowski.

To whom it may concern:

As a referee designated by the Academic Council for Mathematics & Computer and Information Sciences, University of Warsaw, I give the following evaluation of the thesis in question. I declare that there is no conflict of interest between me and either the PhD candidate or his advisor.

Overview

The thesis of M. Brzozowski deals with various estimates for *martingale transforms*, which may be seen as probabilistic counterparts of singular integrals from mathematical analysis. Both discrete-time and continuous-time versions of these estimates are considered, as well as versions involving either the martingales themselves or their maximal functions. The common framework that the author applies to deal with these problems is the so-called *Bellman function* method. This method is so central to the thesis that it could have deserved to appear in the title.

The particular estimates under consideration are *weighted norm inequalities*, mostly involving either strong or weak Lebesgue L^p norms with a weight function w . It is worth a separate mention that the established but slightly less studied “weak L^∞ ” space of De Vore and Sharpley is also treated. The weights that are relevant to the studied problems (like so many others) are those satisfying a celebrated Muckenhoupt condition, either A_p or A_∞ .

By “sharp weighted inequalities” (as in the title), one understands estimates that establish the best possible dependence on the “Muckenhoupt constant” of the weight, in the sense that any hypothetical improvement of this dependence is no longer valid.

The analogues of these sharp weighted inequalities for singular integrals have been the topic of extensive study during this century, with a particularly high

intensity in the last 10 years. While some individual cases of the martingale estimates (particularly for the simplest possible case of dyadic martingales) were obtained alongside with these analytic results, a more systematic development of the probabilistic counterpart of this theory is rather more recent. Some key works to mention in this direction are papers by C. Thiele, S. Treil and A. Volberg (Adv. Math. 2015), M. T. Lacey (Israel J. Math. 2017), and K. Domelevo and S. Petermichl (Ann. Probab. 2019), all included in the bibliography of the thesis (along with many others). Most of the mentioned names are top-rate mathematicians (for instance, several of them have delivered an invited talk at the International Congress of Mathematicians, which is regarded as a significant distinction), and this clearly shows the high international relevance of the topic.

The contributions of this thesis are roughly of the following two types:

1. Provide a unified approach to the various different inequalities by a systematic application of the Bellman function method.
2. Make use of this approach to obtain several extensions and variations of the known results. In particular, maximal function variants of some estimates are achieved with relatively minor modifications of the Bellman function for the non-maximal version.

Methods

As said, the main mathematical tool employed by the author is the so-called Bellman function method. The background for the application of this method is explained in a detailed way that is easy to follow. The real mathematical challenge is then in the construction of the relevant Bellman function for each problem. The Bellman functions appearing in the main contributions of this work have between four and six real variables. Although they are given by explicit expressions consisting of piecewise elementary functions, in practise the computations with these functions are quite technical and involved. The verification that these functions actually satisfy the three required conditions (an initial condition, a majorization property, and a concavity-type property, which is usually the most difficult of the three) is an arduous task of several pages of technical computations, and one can only imagine how many more pages of calculations it has taken the author to come up with the formulas for these functions in the first place. I confess that I have not gone through all the details of all these computations, but the several steps that I checked here and there were all correct.

It is clear that during his PhD project, the author has developed quite an impressive mastery of the Bellman function method. Considering the fact that relatively few mathematicians have the same drive for this method, there is a risk that new achievements using the Bellman approach, possibly involving even more complicated functions of even more variables, will not reach the maximal audience that they might deserve, so it might be worthwhile making some suggestions about the organisation of such arguments:

Large portions of the computations are in principle “trivial” (but very tedious by hand), and could probably be carried out by symbolic computer algebra. Although it is certainly a merit that the proofs remain human-verifiable, it might be worthwhile separating

1. those parts of the argument that could be automatised by computer algebra (for instance, exact computations of some determinants), and
2. those parts, where an actual mathematical insight is applied (for instance, replacing a complicated expression by a simpler one in an estimate).

Presentation

Overall, the thesis represents mature mathematical presentation, both when it comes to statements of mathematical results and proofs, and when it comes to the more informal discussion, motivation and historical background provided for these results. The thesis is extremely carefully written, with very few and unimportant typos.

A few minor points of criticism follow:

1. It would have improved readability to list the properties 1° through 3° of the Bellman function (in each section where they appear) in a numbered environment like “Definition 2.4”, and then to refer to this definition number (instead of, say, “property 1°”), when relevant. Since the “Bellman list numbers” 1° through 3° are independent of the other numbered environments in the thesis, it was sometimes difficult to browse back to the correct place, to check these conditions, when they were referred to several pages later.
2. In Theorem 2.3, the function G is defined with unspecified parameters κ, C , and a specific choice for them is only made inside the proof of Lemma 2.4. Nevertheless, these specific values are again used without reference inside the proof of Theorem 2.11. It took me quite a while to figure out where the 1109^2 in the proof of Theorem 2.11 came from!
3. In general, an author should not assume that the reader remembers everything that has been said, but provide a precise reference on where to find the relevant information. For instance, on line –9 of page 35, “Arguing as previously” could be made more reader-friendly by providing a precise pointer to the relevant place in the text.
4. It would improve the readability of large numbers to use a separator between groups of three digits, say 294 400 instead of 294400.

However, these points of criticism are rather minor relative to the overall merits of the presentation.

Use of literature

The bibliography of the thesis consists of 87 items, which gives a rather comprehensive list of different works on the topic and shows the author’s good knowledge of the relevant literature. I was particularly delighted to find a reference to a classic 1957 book of R. Bellman, and a mention of this in the Introduction. Despite the rather widespread use of what is commonly called the Bellman function method, it seems that this original work is seldom mentioned, so that the connection between the method and the name of Bellman has been left a bit mysterious in large parts of the earlier literature. For this reason, I would give

particular credit to the candidate for also finding out and citing this original historical source of his topic.

I could not spot any serious omissions in the bibliography. A couple of minor ones are the following:

1. In Remark 3.2, it might have been worth mentioning the recent work [2].
2. Close to Theorem 3.15, the author could have mentioned the analytic counterpart of this theorem, which is contained in [1, Corollary 12.1].

However, these slight omissions are very minor relative to the overall merits of the author's bibliographic citations.

Evaluation

As discussed in more detail above, the thesis entitled “Sharp weighted inequalities for martingales” by Mr. Michał Brzozowski presents several challenging and original contributions in a timely topic in modern mathematics. The thesis is written in professional style, and it provides appropriate references to the relevant earlier literature.

I deem the thesis as sufficient to grant a PhD.

Sincerely,



Tuomas Hytönen

References

- [1] T. P. Hytönen, M. T. Lacey, H. Martikainen, T. Orponen, M. C. Reguera, E. T. Sawyer, and I. Uriarte-Tuero. Weak and strong type estimates for maximal truncations of Calderón-Zygmund operators on A_p weighted spaces. *J. Anal. Math.*, 118(1):177–220, 2012.
- [2] A. K. Lerner, F. Nazarov, and S. Ombrosi. On the sharp upper bound related to the weak Muckenhoupt–Wheeden conjecture. *Anal. PDE*, 13(6):1939–1954, 2020.