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Report on Marcin Wrochna thesis « The topology of solution spaces of combinatorial problems »

The main notion studied by Marcin Wrochna in his thesis is graph homomorphisms, a generalization of colorings. This is a well-studied area which is still mysterious due to a famous conjecture from Hedetniemi : Is the chromatic number chi(GxH) at least the minimum of chi(G) and chi(H) ? This could just be another question of graph theory but it appears that in terms of homomorphisms, GxH is the largest graph (core) mapping to G and H, and thus the best choice for a definition of « intersection » of G and H. The graph GxH should then capture all homomorphism-type properties both of G and H. If Hedetniemi's conjecture would be true, this would mean that chromatic number is closed « under intersection », hence suggesting that there is a unified reason why a graph is not k-colorable. This is against the general feeling since one can construct very different families of graphs with high chromatic number. I want here to highlight this point : *the questions studied in this thesis are central in graph theory and of prime importance concerning graph coloring.*

The general goal of this thesis is to adopt a topological point of view to adress these homomorphism questions. This is not surprising since the original proof of El-Zahar and Sauer 30 years ago is based on a parity argument and hence suggests that topology is the right approach. However, Mr Wrochna is the first to systematically use topology to investigate these questions. His thesis document is a milestone in this field which will certainly inspire further researchers.

The main contribution of Marcin Wrochna include algorithms to find shortest recoloring, new classes of multiplicative graphs (circular cliques and square-free graphs), and topological corollaries of Hedetniemi's conjecture (multiplicativity of spheres for instance).

Let me give a brief overview of the thesis :

In Chapter I, the basic notions are gently introduced, and even if this part could perhaps be a little bit more expanded for the unfamiliar reader, each sentence is carefully chosen to keep a right balance between a concise exposition and a clear introduction. I found this dense yet readable first section very nice to read and an excellent recap on both graph product, exponentiation and classical complex coming from combinatorial structures.

In Chapter 2, the algorithmic aspect of graph recoloring is investigated. Namely, it is showed that finding the shortest path between two H-coloring of some graph G can be computed in polynomial time when H is square-free. This is a far-reaching generalization of what was known for 3-coloring and illustrates how the topological point of view is useful here. It is surprising that turning the problem into homotopy of maps (continuous coloring along edges) allows to fully capture the complexity of finding a shortest path in the discrete setting.

In Chapter 3, the main result is (again) a far-reaching and beautiful generalization of the well-known result that the triangle is multiplicative (El-Zahar and Sauer proof of Hedetniemi for chromatic number 4). Marcin Wrochna shows here that whenever the graph H has no square, then if GxG' maps to H, one of G,G' maps to H (H is *multiplicative*). Again, map recolorings play a crucial role here, and the original coloring can be then iteratively pushed to either G,G' or a cycle. This chapter is particularly appealing to the reader familiar with the classical proof of El-Zahar and Sauer since he will recognize the odd-cycle based argument in a much more general setting.

In Chapter 4, some topological corollaries, deriving from the supposed validity of Hedetniemi, are shown (for instance the beautiful conjecture 1.6 asserting that spheres are multiplicative). The key of the proof is the functor omega_k, which acts as an inverse power and the main point is that the Box complex of G and omega_k(G) are Z2-equivalent. New classes of multiplicative graphs are also provided : omega_3(H), where H has girth at least 12. This section is particularly interesting in the sense that, unlike the previous sections where the argument seems to reach its limit, new and exciting directions of research are proposed here.

Concerning the document itself, I have found the manuscript very well written, each definition and property being carefully introduced at the right time. The proofs are sometimes quite technical and involved, but their arguments are often sketched in advance to help the reader. The illustrations are also extremely helpful to follow the exposition and to guide the reader. I could hardly find any inaccuracy or sloppy statement in the whole thesis, which I will keep at hand for further reading in the future.

The overall comment on this thesis is that Marcin Wrochna has done an impressive work and has thoroughly extracted what can be derived from topological methods. In some sense, it seems that this thesis goes as far as possible in Hedetniemi's question, leaving only open the next step beyond homotopy of paths and cycles which would open the way for the general case.

In conclusion, Marcin Wrochna is an excellent work and his doctoral dissertation can be defended without any revision.

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