

# UPDATED REPORT ON ‘RESOLUTIONS OF QUOTIENT SINGULARITIES AND THEIR COX RINGS’ BY M. GRAB

## OVERVIEW

The PhD thesis of Maksymilian Grab studies in detail the Cox ring for several classes of varieties  $X$  that arise as crepant resolutions of a quotient singularity  $\mathbb{C}^n/G$  where  $G$  is a finite subgroup of  $\mathrm{GL}(n, \mathbb{C})$ .

The importance of the Cox ring was highlighted by the celebrated work of Hu and Keel [53] which establishes that the birational geometry of a  $\mathbb{Q}$ -factorial, projective variety with a finitely-generated and free class group can be understood completely and simply in terms of VGIT for an ambient toric variety precisely when the Cox ring is finitely generated. Our understanding of Cox rings was enhanced considerably over the next decade by Hausen and his collaborators using techniques of invariant theory and combinatorics; much of this work appears in the wonderful book [4]. Crucially, their programme meant that one no longer had to assume that  $X$  was projective in order to obtain strong results linking the Cox ring to birational geometry. More recently, the work of Andreatta and Wiśniewski [2] suggested that Cox rings should also be studied in the relative setting. The work of this thesis builds on this observation to study systematically the Cox rings of smooth varieties that are projective over a suitably nice affine base; in this case, the base is normally taken to be an affine quotient singularity.

After Chapter 2 introduces material on resolutions of quotient singularities (in low dimensions or in the case of a symplectic quotient), the first main result appears in Chapter 3 (see Theorem 3.4.7), where the work of Hu and Keel is extended to the relative setting. The author generalises the approach taken by Hausen et. al. [4], and this result provides a firm foundation for the rest of the thesis. Chapter 4 introduces a criterion (see Theorem 4.1.14 and Prop 4.1.19) to check whether a given set of elements generates the Cox ring and hence whether one can compute explicitly the movable cone of a minimal model of a Gorenstein affine quotient singularity (in fact the result holds more generally) and hence understand the birational geometry of any crepant resolution (if one exists). The first applications appear in Chapter 5 (which contains some of the author’s joint work with Donten–Bury [32]) where the Cox ring of the  $G$ -Hilbert scheme is studied for certain finite subgroups of  $\mathrm{SL}(3, \mathbb{C})$  arising from a finite subgroup of  $\mathrm{GL}(2, \mathbb{C})$ : complete results for dihedral subgroups are established, and some individual examples from other classes are also computed in detail. Following a brief interlude computing the equivariant Euler characteristic of a torus-equivariant line bundle in Chapter 6, the author concludes in Chapter 7 by computing explicitly the Cox ring, the movable cone and the geometry of GIT quotients for crepant resolutions of symplectic quotient singularities in dimension four (admitting a 2-torus action); some, but not all, of this content is joint work with Donten–Bury [31].

## RECOMMENDATION

This is a wonderful PhD thesis. I am very impressed, both in the clear discussion of the theoretical aspects and in the genuinely impressive computational aspects of the work. The exposition is almost universally very clear, and the depth of understanding that is displayed throughout is impressive.

The early chapters were a pleasure to read, and I like very much the elegance of the statement of Theorem 3.4.10. I find the computations presented in Chapter 5 to be very interesting indeed. In fact, I'm embarrassed to note that before reading this thesis in detail, I had not appreciated the importance of Example 5.4.1 and that from Section 5.5, but I am very happy to know about these examples now. Put simply, this work provides a wonderful testing ground for conjectures on crepant resolutions of threefold quotients beyond the settings where we already understand the full picture, namely the abelian case and the case of fibre dimension one.

Finally, the results from Chapter 7 on crepant resolutions of symplectic quotients show how a given subring of the Cox ring can be shown to coincide with the Cox ring itself by computing appropriate Hilbert series; this provides a complete understanding of the birational geometry in these examples. I'm genuinely excited by this approach as it opens up the possibility of tackling some (as yet unpublished) conjectures of mine about Cox rings of various classes of quiver variety. Put simply, not only are the results of this chapter genuinely interesting in themselves, but they may well be more widely applicable. To sum up, it has been a pleasure to examine this lovely thesis, and I look forward to discussing a potential new project with the author!

*More formally, I confirm that this thesis in its current form is sufficient to grant a PhD. In addition, I am very happy to recommend that the PhD is granted 'with an honorary distinction'.*

## COMMENT FOR THE AUTHOR

I was happy to see that the author notes correctly that three of the resolutions in Example 2.2.25 are non-isomorphic only when regarded as varieties over  $Y$ .