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Evaluation report on Doctoral Thesis of Maja Szlenk

Applications of various compactness methods in the context of compressible fluid mechanics

The Thesis summarizes three results of the author, specifically

- Weak solutions for the Stokes system for compressible fluids with general pressure, published in *Journal of Differential Equations*, 2022;
- Weak solutions for the compressible non–Newtonian Stokes system with unbounded divergence, together with M. Pokorný published in *Mathematical Methods in the Applied Sciences*, 2023;
- An attraction–repulsion system in the framework of compressible viscous flows, still unpublished work.

By "Stokes system", the author means a quasi–static approximation of the Navier–Stokes system, where the time derivative as well as the convective term are omitted in the momentum equation. The attraction–repulsion system is inspired by models of collective behaviour.

I. Comments on the results in Chapter 2

A quasi–static system of equations

$$\partial_t \rho + \mathrm{div}(\rho \vec{u}) = 0,$$

$$-\mu \Delta \vec{u} - \lambda \nabla \mathrm{div} \vec{u} + \nabla p(\rho) = 0,$$

supplemented with periodic boundary conditions is studied. The existence and uniqueness is established for a bounded and non–negative initial density. The pressure state equation is rather general allowing non–monotone dependence on the density.

With the ansatz curl$\vec{u} = 0$, the problem is transformed to a simple transport equation that is solved by the Lagrangian method. The main novelty with respect to the existing results is boundedness of the density and uniqueness of solutions based on careful analysis of the transport (continuity) equation.

The proof is based on a Lagrangian–Eulerian transformation in the style of Crippa and De Lellis stability and an improvement of a logarithmic inequality for BMO functions by Mucha and Rusin. The technique of the proof is an alternative to a more complicated approach based on Lions' theory or its recent modification by Bresch and Jabin.

Questions:

1. In Section 2.1, it is claimed that the pressure p must be unbounded. This hypothesis is however missing in Theorem 2.1
2. Condition (2.3) could be possibly relaxed as equation (2.2) yields uniform boundedness of the density for a larger class of functions.

II. Comments on the results in Chapter 3

A "Stokes" like system similar to that one investigated in Chapter 2 is considered, with the difference that the viscous stress is a non–linear function of the velocity gradient. More specifically, the viscous stress corresponds to a power law fluid.

The proof uses the technique of ref. [45], however, due to the simplified character of the equations, avoids the uniform bounds on div$\vec{u}$ enforced in [45] by rather non–standard singular form of the bulk viscosity. The nonlinear viscosity can be singular at zero allowing fluids of Herschel-Bulkley rheology.

Questions:

- Is it necessary that the test function ψ in Definition 3.1 is non–zero up to the initial time?
- Would it be possible to show uniqueness of solutions?
- Remark 3.9. Can one characterize the limit by a subdifferential of some convex functional ?
- Would it be possible to include the time derivative in the momentum equation still omitting the non–linear convective term?

III. Comments on the results in Chapter 4

This chapter is probably the most interesting part of the Thesis. A pressureless Navier–Stokes system with a non–local forcing term is analyzed by a non–trivial adaptation of the method proposed by Vasseur and Yu. The non–local term is characteristic for attractive-repulsive forces arising in models of collective behaviour.

The extremely technical proof of the existence of weak solutions is carefully carried out in full detail and is almost 50 pages long. Here, it is worth noting that the approach of Vasseur and Yu is quite complex and requires mastering various new techniques – estimates.

To the best of my knowledge, a similar system has not been considered before in the available literature and the work has a potential to become a useful reference material for this type of problems.

Questions:

- Could some form of the pressure be included?
- Could the proof be carried over for constant viscosity and with a suitable pressure term?

IV. Conclusion

The Thesis represents a significant amount of work. The candidate mastered different new techniques used in the theory of compressible fluid flows and applied them in a creative way to obtain new results. Two chapters have already been published in high level international journals, the publication of the third one is in preparation. The Thesis is clearly written, the proofs are elaborated with great care. The results are new, non–trivial, and, in particular those obtained in Chapter 4, highly original.

On the basis of the present Thesis, I am happy to recommend the candidate Maja Szlenk to be granted the academic title PhD.

Remarks

- page 3, line - velocity instead of density
- In contrast with what is claimed in the introduction, the existence of global in time solutions to the compressible Euler system is known. There are numerous results in various classes of weak solutions obtained by De Lellis, Székelihidi, Chiodaroli, Kreml and many others... They have been obtained recently by the application of the method known as convex integration.

- Chapter 2, page 12. It is not completely true that the condition curl$\vec{u} = 0$ implies the integral mean of $\vec{u}$ vanishes. A counterexample is a constant vector field $\vec{u}$.
- It is not very clear from hypothesis (3.4) if both μ_0 and λ or only λ should be of order $1/z$.
- Page 31, remarks concerning problem (3.9). The problem is ill posed in the sense there exists infinitely many solutions. The "mere" existence is therefore guaranteed by the method of convex integration as long as the energy inequality is not required.
- Page 46. Even in the case of Poisson kernel, the problem does not coincide with the Navier–Stokes–Poisson system as the pressure is absent.

In Praha, 29 April 2024

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