Review of Mr. Maciej Markiewicz’s Thesis

The doctoral thesis of Mr. Maciej Markiewicz, submitted to the University of Warsaw, studies an invariant of links in the 3-space which is a multi-variable version of the Levine-Tristram signature function.

In the literature, the Levine-Tristram signature function was first introduced to study knots and played a key role in the foundational work of Levine on knot concordance. Since then, the Levine-Tristram signature function has been used in various contexts as a primary concordance invariant. The main subject of this thesis is a natural generalization of the Levine-Tristram signature function of links, i.e., embeddings of a disjoint union of circles. In the case of knots, which have a single component, the signature function has one variable. Compared with this, for a multi-component link, the signature function is naturally generalized to a multi-variable function, which is the main subject of this thesis.

In particular, the author focuses on the limit of the multi-variable Levine-Tristram signature function as some of the variables approach one. This is very naturally motivated, especially by a well-known behavior of the multi-variable Alexander polynomial of links. Roughly speaking, the multi-variable Alexander polynomial measures the first homology of the universal abelian covering space of the link complement, while the signature function concerns further information on the duality on the first homology of the covering space. A classical fundamental result of Torres relates the multi-variable Alexander polynomial with some variables set to one to the multi-variable Alexander polynomial of an associated sublink. This naturally raises an analogous question for the multi-variable signature function: can we relate the limit of the multi-variable signature function as some variables approach one (or the evaluation when some variable is set to one) to the multi-variable signature function of an associated sublink?

The main results of this thesis give an answer to this question. In particular, Theorem 1.1 and Theorem 1.2 provide an explicit relationship. Also, as Theorem 1.3, the author proves similar results for another closely related invariant called the nullity, which measures the extent of the degeneracy of the duality. In addition, the author addresses the case of colored links, for which multi-components of a given link are allowed to be associated with the same variable.

The author’s approach, especially the proofs of the main results, are also
closely related to earlier research. Briefly, there are two key methods of the
study of the signature function of knots in the literature: one is essentially
3-dimensional, often developed using Seifert surfaces (and an associated
matrix called the Seifert matrix), and another is 4-dimensional, using the
middle-dimensional duality of an appropriate bounding 4-manifold. (For
links, the 3D method requires a generalization of a Seifert surface, which
is called a C-complex, introduced by Cooper). The thesis under review
builds on both approaches, taking advantage of each of them, to push it
forward as far as possible. In particular, Theorem 1.1 and Theorem 1.2 are
the outcomes of these two approaches, respectively. This shows that the
author has a thorough understanding of these methods at a good working
knowledge level.

To summarize, the thesis of Mr. Maciej Markiewicz studies an interesting
problem on the signature function of links, with a strong motivation from
erlier results of fundamental importance in knot theory, and proves in-
triguing results that present a satisfactory answer to the questions. The
main techniques used in this thesis are closely related to earlier important
approaches in a stimulating way.

Based on this, I gladly deem the thesis sufficient to grant a Ph.D.

I attach a separate list of detailed comments, for the author to improve
some parts of the exposition. I do not ask the thesis to be returned to me
(nor other reviewers), because all those comments are minor (e.g. simple
typos) and can be easily addressed.

Jae Choon Cha
Details Comments on Mr. Maciej Markiewicz’s Thesis

1. Page 4, line 9: “...colored links, that is links where...”
   Use “that is,” (with a comma), or “which are”

2. Page 5, the 3rd displayed equation
   “...” \rightarrow “...” (\cdots)

3. Page 5, line 22:
   impose \rightarrow imposes

4. Page 5, line 32:
   sum \rightarrow the sum

5. Page 7, line 9:
   manifolds \rightarrow manifold

6. Page 25, line 11:
   go over go \rightarrow go over

7. Page 25, line -10:
   matrices divisible \rightarrow matrices are divisible

8. Page 25, end of page:
   Add a punctuation symbol (period or colon).

9. Page 26, line 10:
   Both “resp.” and “respectively” appear in this manuscript — use them consistently.

10. Page 30, line 3:
    entire ring \rightarrow the entire ring

11. Page 33, line 1:
    strictly positive real scalar \rightarrow a strictly positive real scalar

12. Page 36, line 1:
    Remove one of the two occurrences of “also.”

13. Page 63, line 5:
    of link \( L_i \) component \rightarrow of the link component \( L_i \)

14. Page 63, line -3:
    expressions \rightarrow elements

15. Page 65, line 7:
    depend \rightarrow depends
    coincide for \( s = 0, 1 \) \rightarrow the dimensions for \( s = 0, 1 \) coincide

16. Page 65, line 13:
    \( s = 0, 1 \) then \rightarrow \( s = 0, 1 \), (add a comma and remove “then”)