Kraków, 7th of November 2023

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Referee report on the corrected PhD thesis by mgr Maciej Gałązka
"Secant varieties, Waring rank and generalizations from algebraic geometry viewpoint"

The scientific content of the corrected PhD thesis by Mr. Maciej Gałązka does not substantially differ from the original version that I refereed on the 9th of August 2023.

Mgr Maciej Gałązka did improve the exposition, following the comments by Prof. Maria Chiara Brambilla, and he did remove some minor missprints from Chapter 3 (e.g. Theorem 3.30 in new version of the thesis is more precise than its predecessor - Theorem 3.31 in the former version). Consequently, **I find the current version of the thesis even better than the previous one** and I can repeat the positive conclusion of my previous review to asses the final version of the PhD Thesis by Mr. Maciej Gałązka, with a minor change in my judgement on the exposition:

1) **The thesis under review contains original solutions to non-trivial mathematical problems and deals with questions that are subject of intensive research.** The presented results are original, interesting and could be useful for other researchers. In particular, my claim can be partially substantiated by the fact that most of the results discussed in the thesis were published by Gałązka in two papers in good journals and a joint paper (with Mańdziuk and Rupniewski) in a very good journal (with equal contributions of each of the authors, according to the authors' statement in Foundations of Computational Mathematics). It should be pointed out that the above papers by M. Gałązka have been cited by other researchers.

2) The reasonings presented in the thesis show that the author posesses sound geometric intuition, a good command of various techniques of algebraic geometry (toric geometry, deformation theory) and algebra (e.g. Groebner bases, divided power structures). In particular, after reading the thesis under review I have no doubt that **mgr Maciej Gałązka possesses general theoretic knowledge of the field in which he carried out the research to write the thesis.**

3) In my opinion, originality and technical complexity of the reasonings presented in the thesis demonstrates that **mgr Maciej Gałązka is able to carry out independent research in algebraic geometry** and that he can be creative.

4) The presentation of the results in the thesis under review is good and confirms Mr. Gałązka's ability to explain his ideas in a precise manner. The thesis is relatively long (more than 120 pages) and quite technical, but Mgr Gałązka is able to present

fairly complicated reasonings in a clear way.

Conclusion: In my opinion, the PhD thesis "Secant varieties, Waring rank and generalizations from algebraic geometry viewpoint" submitted by mgr Maciej Gałązka fulfills all requirements posed on a PhD thesis. Consequently, I recommend it for the defense in front of the appropriate committee.

Conclusion in Polish: Moim zdaniem rozprawa doktorska "Secant varieties, Waring rank and generalizations from algebraic geometry viewpoint" Pana magistra Macieja Gałązki spełnia wymogi dotyczące rozprawy doktorskiej zawarte w Ustawie "Prawo o szkolnictwie wyższym i nauce" z dnia 20 lipca 2018 roku. Wnioskuję o dopuszczenie Pana magistra Macieja Gałązki do dalszych etapów postępowania w sprawie nadania stopnia doktora.

Sławomir Rams

(Sławomir Rams)

Kraków, 9th of August 2023

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Referee report on the PhD thesis by mgr Maciej Gałązka
"Secant varieties, Waring rank and generalizations from algebraic geometry viewpoint"

The PhD thesis by Mr. Maciej Gałązka is devoted to various notions of rank of a homogenous polynomial (Waring rank, border rank, cactus rank) and their generalizations. One can give geometric interpretations of the above integers in the language of the so-called secant varieties and cactus varieties, so the latter varieties play prominent role in the thesis under review. In particular, the choice of geometric language enables the author to obtain more general results, e.g. bounds on the number of summands in decompositions of certain polynomials into sums of products of powers of linear forms ($r_{\mathbb{P}^1\times\mathbb{P}^1}(\cdot)$ in the notation of the thesis - see p. 10).

The thesis of Mr. Gałązka consists of six chapters. Below I briefly discuss their contents.

In Chapter 1 the author recalls two fundamental questions on polynomial/tensor decomposition that has motivated researchers for a long time (Problems 1.1, 1.2) and (very concisely) discusses the current state of knowledge. Then, he proceeds to formulate various results that are proven in the thesis. Mgr Gałązka states his bounds on the value of $r_{\mathbb{P}^1\times\mathbb{P}^1}(x^k y^l z^m w^n)$ (Theorem 1.6), gives some equations of the r-th cactus variety of a projective variety embedded by the complete linear system of a very ample line bundle (Theorem 1.13), states a new upper bound on cactus ranks of tensors in $\mathrm{Sym}^{d_1}\mathbb{C}^{n_1+1}\otimes\ldots\otimes\mathrm{Sym}^{d_k}\mathbb{C}^{n_k+1}$ (Proposition 1.15). The latter result is stronger than the bound given by Ballico, Bernardi and Gesmundo in 2017. Finally, after short discussion of Hilbert schemes and importance of Gorenstein locus therein mgr Gałązka formulates a very interesting result of his thesis - Theorem 1.17, which gives precise description of components of the cactus variety $\kappa_{14}(v_d(\mathbb{P}^6))$ (actually, it is a corollary of a far more general Theorem 6.1, and it reappears in Chapter 6 as Corollary 6.4). It should be mentioned that certain basic notions that appear in formulations of the above results are defined in Chapter 1, e.g. the X-rank, the secant variety, the cactus rank, the cactus variety, Segre-Veronese embeddings, which is very helpful to the reader.

A well-known formulation of Apolarity Lemma (see e.g. the book by Iarobbino and Kanev) deals with projective spaces and uses the language of homogenous ideals. Many considerations of Gałązka's thesis are carried out in the language of toric varieties and their Cox rings, which is a natural generalization of the classical projective set-up. That is why the second chapter of the thesis is devoted to toric

S.R.

varieties. §.2.1 of the thesis focuses on the notion of saturated ideal. In remainder of Chapter 2 Mr. Gałązka collects certain properties of homogenization and dehomogenization, formulates the definition of divided power structures and the apolarity action, gives a formula for the value of the map defined by a complete linear system in a point (Proposition 2.25 - to be used later in §.5.3). Finally, Gałązka presents more precise technical results on dehomogenization and homogenization in the case of multiprojective spaces and Veronese embeddings.

In the first part of Chapter 3 the author formulates and proves two versions of Apolarity Lemma - one for projective varieties embedded by a complete linear system of a very ample line bundle (Proposition 3.6) and the other for toric varieties (Theorem 3.9). Both statements appeared in Gałązka's paper in Mathematische Nachrichten. Then, he proves Theorems 1.13, 1.14 and gives a serious of examples that demonstrate the fact that the equations coming from vector bundles fail to describe certain secant varieties (Corollary 3.19). Subsequently, Gałązka proves two bounds on cactus rank of certain finite dimensional spaces (Theorem 3.31) and forms (Theorem 3.32), that play important role in the sequel. Here the arguments heavily depend on the preparations in Chapter 2 and impose some demands on the patience of the reader. Finally, in the last subsection of Chapter 3 the author presents the proof of Proposition 1.15, that is based on Theorem 3.31, and gives some explicit examples that illustrate the power of his results. As in previous chapters, Gałązka also tries to help the reader by recalling some basic properties of secant varieties (e.g. Terracini's Lemma) and various useful facts on the border rank and the border cactus rank.

Chapter 4 of the thesis under review is devoted to some applications of Hilbert schemes. At first Gałązka recalls two theorems on the number of irreducible components of the Gorenstein locus $\mathcal{H}ilb_r^{Gor}(\mathbb{A}^n)$ for $r \leq 14$, the whole Hilbert scheme $\mathcal{H}ilb_s(\mathbb{A}^n)$ for $s \leq 8$, and gives formulae for the dimensions of tangent spaces to some components of the above schemes at non-smoothable points (Lemmata 4.4-4.5). The latter results play important role in proofs of Theorems 6.6, 6.7 in Chapter 6. Furthermore, for $n = n_1 + \ldots + n_k$, he compares the number of components of $\mathcal{H}ilb_r(\mathbb{A}^n)$ (resp. $\mathcal{H}ilb_r^{Gor}(\mathbb{A}^n)$) and $\mathcal{H}ilb_r(\mathbb{P}^{n_1} \times \ldots \mathbb{P}^{n_k})$ (resp. $\mathcal{H}ilb_r^{Gor}(\mathbb{P}^{n_1} \times \ldots \mathbb{P}^{n_k})$) - see Proposition 4.3. Then the author uses the rational map that assigns to a scheme its projective linear span to show that the 14-th cactus variety $\kappa_{14}(v_d(\mathbb{P}^n))$ of the Segre-Veronese embedding has at most two irreducible components, one of which could be the secant variety $\sigma_{14}(v_d(\mathbb{P}^n))$. He also shows the inequality

$$\dim(\eta_{14}(v_d(\mathbb{P}^n))) \leq (14n + 5), \tag{1}$$

where $\eta_{14}(v_d(\mathbb{P}^n))$ denotes the second potential component of $\kappa_{14}(v_d(\mathbb{P}^n))$. Then, analogous analysis is carried out for the Grassman cactus variety $\kappa_{8,3}(v_d(\mathbb{P}^n))$, with the bound $\dim(\eta_{8,3}(v_d(\mathbb{P}^n))) \leq (8n + 8)$. Subsequently, Gałązka constructs a morphism from certain locally closed reduced subscheme of the l-th Grassmanian of the degree-at-most-m part of the graded dual of the ring $\Bbbk[\alpha_1, \ldots, \alpha_n]$ to the Hilbert scheme $\mathcal{H}ilb_r(\mathbb{A}^n)$ (Theorem 4.8). The proof is based on study of the interplay between the apolarity action and tensor products. Next, after defining the triangle operator that gives a uniform way of treating forms divisible by some powers of linear forms, Gałązka finds an irreducible $(14n+6+k)$-dimensional subset C of

$$\prod_{i=1}^{k} V_i^* \times T_{3,\dots,3} \tag{2}$$

and its dense subset D that consists of sequences $(z_1,\dots,z_k,P)$ such that P defines (via appropriate dehomogenization, triangle operator and Apolar$(\cdot)$ - see p. 86) a scheme away from $\mathcal{H}ilb_{14}^{Gor,sm}(\mathbb{A}^n)$ (Lemma 4.20). Finally, Gałązka carries out analogous constructions for $\mathcal{H}ilb_8^{sm}(\mathbb{A}^n)$, i.e. he constructs an irreducible $(8n+9)$-dimensional subset C of $T_1 \times \mathrm{Gr}(3,T_2)$ and its dense subset whose all elements in similar manner define schemes away from $\mathcal{H}ilb_8^{sm}(\mathbb{A}^n)$.

Chapter 5 contains calculations of the rank, the cactus rank and the border rank of some monomials for several toric surfaces (i.e. $\mathbb{P}^1 \times \mathbb{P}^1$, Hirzebruch surface $\mathbb{F}_1$, $\mathbb{P}(1,1,4)$, a fake weighted projective plane) embedded into projective spaces. In particular, in one of the examples the border rank is strictly smaller than the smoothable rank (Remark 5.4).

Chapter 6 is devoted to distinguishing certain secant varieties from cactus varieties. At first mgr Gałązka states main results. Each of Theorems 6.1-6.3 consists of two parts; the first part is the claim that a cactus variety $\kappa_\cdot(\cdot)$ has exactly two irreducible components: the secant variety $\sigma_\cdot(\cdot)$, and the component defined as $\eta_\cdot(\cdot)$ in Chapter 4, whereas the second part is a characterization of a dense subset of $\eta_\cdot(\cdot)$. The assumptions on the number of variables and the (multi)degree are very weak - e.g. $n \geq 6$ and $d \geq 5$ in Theorem 6.1 (with discussion of initial cases in §.6.1). The arguments of the thesis give also a method to determine whether a class $[G] \in \kappa_{14}(v_d(\mathbb{P}T_1^*))$ (resp. $[V] \in \kappa_{8,3}(v_d(\mathbb{P}T_1^*))$) belongs to the secant variety $\sigma_{14}(v_d(\mathbb{P}T_1^*))$ (resp. $\sigma_{8,3}(v_d(\mathbb{P}T_1^*))$) - see Theorem 6.6 (resp. Theorem 6.7). Theorems 6.1-6.3 demonstrate the fundamental merit of the thesis under review: *Mr. Gałązka uses fairly abstract and modern techniques of algebraic geometry and algebra to obtain results that answer natural and important questions.* As an example one can quote Corollary 6.4: for $d \geq 5$ the cactus variety $\kappa_{14}(v_d(\mathbb{P}^6))$ has two irreducible components: the secant variety and the variety $\eta_{14}(v_d(\mathbb{P}^6))$ which consists of degree-d forms that are divisible by the $(d-3)$-rd power of a linear form.
The rest of the last chapter of the thesis is mainly devoted to proofs of the above results. For example, in order to show Theorem 6.1 the author at first applies Theorem 3.32 to prove that

$$[z_0^{d-3}F] \in \eta_{14}(v_d(\mathbb{P}T_1^*)) \setminus \sigma_{14}(v_d(\mathbb{P}T_1^*)) \tag{3}$$

where z_0 forms part of a basis of T_1^*, Apolar$((F|_{z_0=1})^{\blacktriangledown d})$ has Hilbert function $(1,6,6,1)$ and [Spec(Apolar$((F|_{z_0=1})^{\blacktriangledown d})$))] does not belong to $\mathcal{H}ilb_{14}^{Gor,sm}(\mathbb{A}^n)$. Then Proposition 6.12 yields that the map

$$\psi : (z_0, F) \mapsto [z_0^{d-3}F]$$

sends all points from the subset D of (2) (for $k = 1$) to $\eta_{14}(v_d(\mathbb{P}T_1^*))$. By density of D in C, the same holds for all elements of the $(14n + 7)$-dimensional set C. Since the fibers of ψ are 2-dimensional, we obtain

$$(14n + 5) = \dim(\overline{\psi(C)}) \leq \dim(\eta_{14}(v_d(\mathbb{P}T_1^*)))\,.$$

Now, inequality (1) implies that $\eta_{14}(v_d(\mathbb{P}T_1^*))$ is the closure of the set of classes $[z_0^{d-3}F]$, as claimed in Theorem 6.1(ii). Then mgr Gałązka applies Proposition 6.12 (see (3)) to show that the secant variety cannot be the unique component of the cactus variety. Finally, it is fairly obvious that

$$\kappa_{14}(v_d(\mathbb{P}T_1^*)) \neq \eta_{14}(v_d(\mathbb{P}T_1^*)\,,$$

and the proof of Theorem 6.1 is complete.
The proofs of Theorems 6.2, 6.3 are more involved but they follow the same pattern: appropriate choice of a dense subset of $\eta.(\cdot)$, combined with a dimension computation, yields the required result.
In order to prove Theorem 6.6, the author shows that classes $[z_0^{d-2}Q]$ belong to the secant variety (Proposition 6.13), whereas each class in cactus variety that does not belong to the secant variety can be represented as $[z_0^{d-3}F]$ (see Lemma 6.14), where the pair (F, z_0) satisfies extra conditions, that amount to a computation of a Hilbert function, and testing whether a given 0-dimensional scheme belongs to $\mathcal{H}ilb_{14}^{Gor,sm}(\mathbb{A}^n)$. Recall that, by Lemma 4.4, the latter condition can be verified explicitely, because one can compute the dimension of the tangent space. The arguments in the proof of Theorem 6.7 are similar.

After the attempt to describe its content, let me assess the thesis under review:
1) **The thesis under review contains original solutions to non-trivial mathematical problems and deals with questions that are subject of intensive research.** The presented results are original, interesting and could be useful for other researchers. In particular, my claim can be partially substantiated by the fact that most of the results discussed in the thesis were published by Gałązka in two papers in good journals and a joint paper (with Mańdziuk and Rupniewski) in a very good journal (with equal contributions of each of the authors, according to the authors' statement in Foundations of Computational Mathematics). It should be pointed out that the above papers by M. Gałązka have been cited by other researchers.
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Conclusion: In my opinion, the PhD thesis "Secant varieties, Waring rank and generalizations from algebraic geometry viewpoint" submitted by mgr Maciej Gałązka fulfills all requirements posed on a PhD thesis. Consequently, I recommend it for the defense in front of the appropriate committee.

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