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Oslo, October. 2023

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Report on the revised PhD dissertation

"Secant varieties, Waring rank and generalizations from algebraic geometry viewpoint"

by Maciej Galazka

Minor issues in the submitted PhD dissertation raised by two referees have been addressed in the revision. The comments and revisions by Galazka are to the point and appropriate. All issues were minor and insufficient to, in any way, change my general evaluation of the thesis. They clearify some notation and parts of a proof, and they also clearify the relation to some other work in the area.

In conclusion, I stand by my evaluation and find the dissertation sufficient to grant a PhD.

I want to reiterate my assessment from the first report: There is no doubt in my mind, that this dissertation both in content and in form satisfies the requirement for a PhD dissertation in mathematics at any university. I have been on PhD committees at a number of universities both nationally and internationally, and find this dissertation to be clearly above average among confirmed dissertations.

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Sincerely yours,

Kristian Ranestad

Department of Mathematics
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Oslo, June. 2023

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Report on the PhD dissertation

"Secant varieties, Waring rank and generalizations from algebraic geometry viewpoint"

by Maciej Galazka

Secant varieties are fundamental geometric objects of relevance and importance both in theoretical and applied contexts. They are secant varieties of a smaller variety X in an ambient space, here a projective space P. For each natural number r, the closure of the union of linear spaces spanned by r distinct points on X is called the r-th secant variety of X, denoted $\sigma_r(X)$. Finding equations for $\sigma_r(X)$ given equations for X, is in general a completely open problem. One major obstacle is the fact that there are many more finite subschemes of X of length r than r-tuples of points. The closure of the union of linear spaces spanned by subschemes of X of length r is called the r-th cactus variety of X, denoted $\kappa_r(X)$. The varieties $\sigma_r(X)$ and $\kappa_r(X)$ are the main objects of interest in this dissertation. The main questions are:

(I) Given a point p in P, what is the minimal r such that p lies in the span of r points on X, and what is the minimal r such that p lies in the span of a subscheme in X of length r (called rank and cactus rank respectively)?
(II) What are equations for $\sigma_r(X)$ and $\kappa_r(X)$?

These problems are very hard in general, in fact also in many innocuous cases. Almost all known results concern special varieties X. In particular, when X is the d-uple embedding of a projective space. In this case P may be identified with the set of homogeneous polynomials of degree d while X is the subvariety consisting of pure d-th powers of linear forms. In this case a number of special techniques have been introduced in the study of the above two questions, and with some success. The main contribution of this dissertation is to extend these techniques and show consequences for multihomogeneous polynomials where X is a socalled Segre -Veronese variety and more generally a toric variety.
A natural, and classical, approach to (I) and (II) is to consider, for any point p in P, the divisors on X whose span contains p. They form an ideal I_p in the socalled Cox ring, the ring formed by the global sections of all line bundles on X. In general the Cox ring is unmanageable, not finitely generated etc. But if X is a toric variety, the Cox ring is a finitely generated polynomial ring. And if p is not on X, the quotient of the Cox ring by I_p is artinian. Finite subschemes of X whose span in P contain p, are precisely the finite schemes whose ideals in the Cox ring are contained in I_p. This is the classical apolarity lemma when X is the d-uple embedding of a projective space. Galazka extended this lemma to any toric variety in his master thesis. There are many different ideals that define the same scheme, so the correspondence in the apolarity lemma is valid for a particular saturated ideal

of the subscheme. This is one of the concerns that Galazka makes explicit in this dissertation in the case when X is a product of projective spaces. In this case P may be identified with the space of multihomogeneous polynomials, while X is the set of products of powers of linear forms in the different sets of variables, the socalled Segre-Veronese embedding of a product of projective spaces. The apolarity condition may in this case be interpreted using a divided power action of the multihomogeneous polynomial ring on p. The ideal I_p are the forms that annihilate p. Galazka shows that certain dehomogenizations p_i of p have inhomogeneous annihilators whose homogenized ideals Ip_i in I_p define local and finite schemes on X. This idea has been used when X is a d-uple embedding of projective space to give upper bounds for the cactus rank for a general p in P. Galazka provides similar bounds when X is a Segre-Veronese embedding.

Additionally Galazka applies this idea to a problem that generalizes (I): What is the minimal length of a subscheme whose span contains a given subspace L of P. The closure of the set of subspaces L in P that lie in the span of r points, respectively schemes of length r, is called a Grassmann secant variety, respectively a Grassmann cactus variety of X.

Galazka uses the technique of finding local schemes whose ideal lie in I_p to distinguish between the cactus variety and the secant variety, when they are distinct. The distinction depends on the topology of the Hilbert scheme. The Hilbert scheme of finite schemes of length r in affine n-space, is reducible when r>7 and n>3. This is the crucial bounds when the Grassmann secant variety and the Grassmann cactus variety may differ. Galazka shows how they differ for 3-dimensional subspaces L when X is the d-uple embedding of n-space and d>4, r=8 and n>3, by identifying the subspaces L of polynomials that lie in the cactus variety and not the secant variety.
Similarly, the Hilbert scheme of Gorenstein schemes of length r in affine n-space is reducible when r>13 and n>5. Galazka shows how the ordinary secant and cactus varieties of the d-uple embedding of n-space in P differ when n>5 and d>4 and r=14, by identifying the polynomials p in P that lie in the cactus variety and not the secant variety.
In both cases he provides an algorithm that checks whether a given subspace L, respectively a given polynomial p, that lies in the cactus variety, also lies in the secant variety.

To distinguish the secant variety from the cactus variety the optimal tool would be to have the equations of both. Since the secant variety is contained in the cactus variety, one naturally considers on one hand equations that do not distinguish between the two, and on the other hand equations satisfied by the secant variety and not the cactus variety.
The most classical equations for secants varieties of d-uple embeddings are given by the rank condition on certain catalecticant matrices. These are, in fact, equations for the cactus varieties, but in general far from all. Galazka provides two important improvements or generalizations of the catalecticant approach that provide more equations. He calls them, very reasonably, determinantal methods.
A further analysis of the Hilbert scheme as indicated above, is suggested as the way to find equations that vanishes on the components of the cactus variety that are distinct from the secant variety. Galazka in his analysis and first clearifying examples provides the first steps in this approach.

This dissertation gives an excellent overview of methods and results on the topic of secant and cactus varieties. It provides the crucial technical lemmas, some found in the literature, some generalized appropriately by Galazka, and applies them to give a number of new results. Some are cited or rephrased from previous publications by Galazka and coauthors, others are new in this dissertation.
Guided by a nicely written introduction, the reader is made clear throughout, which results are due to whom. The technicalities and notations are sometimes a mouthful, but these are overcome by a pleasant writing style. I could only find an ignorable number of typos, all insignificant.

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There is no doubt in my mind, that this dissertation both in content and in form satisfies the requirement for a PhD dissertation in mathematics at any university. I have been on PhD committees at a number of universities both nationally and internationally, and find this dissertation to be clearly above average among confirmed dissertations.

I therefore deem the thesis clearly sufficient to grant a PhD.

Sincerely yours,

Kristian Ranestad