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Review on the PhD thesis of M. Łukasz Chomienia

M. Łukasz Chomienia's PhD thesis, titled "Partial Differential Equations on low-dimensional structures", investigates parabolic and elliptic problems on low-dimensional structures.

Following the approach of Bouchitté, Buttazzo and Seppecher [Bou97], the author defines a low-dimensional structure in Euclidean space as a measure μ that is singular with respect to the Lebesgue measure of the domain in which it is embedded. Essentially, a low-dimensional structure is a thin set represented by the support of μ, which may exhibit singularities such as corners and junctions.

The PhD thesis aims to explore the relationship between the calculus of variations and the theory of partial differential equations within low-dimensional structures. Although extensively studied in the Euclidean setting, this relationship remains largely unexplored in the context of low-dimensional structures. This PhD thesis addresses this gap, drawing inspiration from the work of Rybka and Zatorska-Goldstein [Ryb20], which marks the first publication dedicated to this subject.

The author utilizes tools, such as μ-tangent spaces, μ-Sobolev spaces and first and second-order μ-tangential derivatives, developed by Bouchitté and Fragalà [Bou03], and other researchers, for studying first and second-order variational problems on low-dimensional structures. These tools are highly technical, and throughout the PhD thesis, the author demonstrates significant expertise in applying and extending them within the context of his research.

The PhD thesis is very well-written and organized into six chapters. Chapter 1 serves as the introduction, efficiently presenting all key results of the PhD thesis and establishing connections with related works on metric measure spaces, the theory of varifolds, and partial differential equations on graphs, thus demonstrating the author's comprehensive understanding of the subject. The subsequent five chapters are based on two papers by the author. The first paper [Cho24], solely authored and published in a reputable mathematical journal, addresses the existence and uniqueness of solutions to parabolic equations on low-dimensional structures. The second paper [Cho23], coauthored with Michał Fabisiak and currently unpublished, explores the higher regularity of solutions to elliptic equations on low-dimensional structures.

Chapters 2 and 3 provide basic concepts and results on tangential calculus with respect to a measure μ, including new characterizations of first- and second-order μ-Sobolev spaces of functions (Chapter 3, §3.3). These concepts and results are essential for the subsequent chapters (Chapters 4, 5 and 6) where the main contributions of the author are presented and developed.

Let $\mu = \sum_{i=1}^{m} \mathscr{H}^{\dim S_i}|_{S_i}$ be a low-domensional structure (Chapter 2, Definition 2.1), where S_i are compact, smooth submanifolds (with boundary) of $\mathbb{R}^3$, which are pairwise transversal and $\dim(S_i) \in \{1, 2\}$.

Let $B = (b_{ij})_{i,j\in\{1,2,3\}}$ be an elliptic and symmetric matrix. The author's first focus lies in investigating the existence and uniqueness of solutions to the parabolic equation $\partial_t u - Lu = 0$ in $\text{supp}(\mu) \times [0, T]$, where $Lu := \text{div}(B\nabla u)$, subject to a Neumann boundary condition and an initial datum (omitted here for simplicity). The problem presents several challenges, which the author successfully addresses.

Firstly, the author establishes the appropriate interpretation of this equation with respect to the measure μ. Building upon the μ-second-order relaxation framework introduced in [Bou03], the author defines the relaxed operator L_μ corresponding to L. This construction, detailed in Chapter 2 (Definitions 2.15, 2.16 and 2.17), leads to the relaxed parabolic equation $\partial_t u - L_\mu u = 0$ $\mu \times \mathscr{L}^1$-a.e. in $\text{supp}(\mu) \times [0, T]$ with $L_\mu u := \sum_{i,j=1}^{3} b_{ij}(\nabla_\mu u^2)_{ij} + \sum_{i=1}^{3}(\nabla_\mu u)_i \text{tr}\nabla_\mu(b_{i1}, b_{i2}, b_{i3})$, subject to a relaxed Neumann boundary condition and an initial datum.

Next, a suitable notion of solution for the relaxed parabolic equation is introduced, based on a semigroup generated by the relaxed operator L_μ (Chapter 2, Definition 2.19, Chapter 4, §4.2).

Finally, establishing the existence and uniqueness of solutions to the relaxed parabolic equation requires adapting techniques from the general theory of differential operators, such as the Hille-Yosida theorem, to the context of low-dimensional structures. A significant challenge is proving the closedness of the relaxed operator L_μ. Particularly, for $B = \text{Id}$, i.e. $L_\mu = \Delta_\mu$, the author successfully resolves these issues (Chapter 4, Theorem 4.1). Notably, a technical result (Chapter 4, Theorem 4.2), the proof of which occupies a substantial part of Chapter 4, resolves the challenge of proving the closedness of the relaxed operator L_μ.

Chapter 4 concentrates on strong-form parabolic problems, examining the existence and uniqueness of solutions through a semigroup approach tailored for low-dimensional structures. This method ensures the existence of solutions within a narrow function space and assumes a highly regular initial datum (Chapter 4, Theorem 4.11).

In contrast, Chapter 5 explores weak-form low-dimensional parabolic problems (as defined in Chapter 2, Definition 2.25) without relying on a semigroup approach. This weak approach broadens the class of admissible initial data and facilitates the study of existence, uniqueness, and regularity of solutions in a more flexible and less complex manner. Several new results within the framework of low-dimensional structures are proved.

First, the author establishes the existence and uniqueness of solutions to weak-form low-dimensional parabolic equations (such as Chapter 5, Theorem 5.4) by adapting the Lions variant of the Lax-Milgram lemma.

Then, by adding regularity assumptions on the initial data and applying operator theory for partial differential equations in Banach spaces to the equation $u' + \mathscr{A}u = f$ in $L^2(0, T, L^2_\mu)$ where the operator $\mathscr{A} : D(\mathscr{A}) \to L^2_\mu$, with $D(\mathscr{A}) \subset \mathring{H}^1_\mu := \{u \in H^1_\mu : \int u d\mu = 0\}$, is defined by $\int(\mathscr{A}u)v d\mu = \int(B\nabla_\mu u, \nabla_\mu v)d\mu$ for all $v \in \mathring{H}^1_\mu$ (Chapter 5, Definition 5.5), the author explores strategies for obtaining more regular solutions (Chapter 5, Proposition 5.7, 5.8 and 5.9).

Additionally, the author establishes a link between parabolic and elliptic problems (Chapter 5, Theorem 5.16) by proving that as $t \to \infty$, solutions of weak-form low-dimensional parabolic problems (as defined in Chapter 2, Definition 2.25) converge to solutions of weak-form low-dimensional elliptic problems (as defined in Chapter 5, Definition 5.15).

The author's second focus concerns the higher regularity of weak solutions to elliptic equations on low-dimensional structures.

In Chapter 6, under an appropriate elliptic setting for B (Chapter 2, Proposition 2.20), the author investigates the higher regularity of weak solutions to the elliptic equation $\operatorname{div}(B\nabla u) = f$ in $\operatorname{supp}(\mu)$ with $f \in L^2_\mu$ such that $\int fg d\mu = 0$ for all $g \in \ker\nabla_\mu$, subject to a Neumann condition. This is considered in the sense introduced in [Ryb20] and formalized in Chapter 2, Definition 2.21. Specifically, a function $u \in H^1_\mu$ such that $\int ug d\mu = 0$ for all $g \in \ker\nabla_\mu$, is called a weak solution if $\int B_\mu \nabla_\mu u \cdot \nabla_\mu \varphi d\mu = \int f\varphi d\mu$ for all $\varphi \in C^1_c(\mathbb{R}^3)$, where B_μ is a relaxation of B as given in Proposition 2.20.

The author's strategy involves applying the difference quotient method within the context of low-dimensional structures. However, this method is specifically designed for functions defined on subsets of $\mathbb{R}^3$ and relies on the Euclidean structure to approximate derivatives and derive estimates. Particularly, the method uses properties such as the translation invariance of Euclidean space, making it challenging to extend to the low-dimensional framework. The author addresses this challenge through a long and technically structured analytic argument, which occupies a substantial portion of Chapter 6, leading to the author's first regularity result which states that weak solutions belong to $H^2_{\mathrm{loc}}(S_i)$ for $1 \leqslant i \leqslant m$ (Chapter 6, Theorem 6.5).

Building on this first theorem, the author establishes a second regularity result, which states that weak solutions are globally continuous (Chapter 6, Theorem 6.8).

Finally, the author emphasizes that within the context of low-dimensional structures, the two previous regularity results are insufficient because they fail to capture any additional connection between higher-order behaviour on different components. He points out that the richness of the μ-second-order framework (Chapter 2, §2.3) provides the most adequate setting for encoding the geometry of low-dimensional structures and establishing correspondences between regularity on various components. The author concretizes this point by proving a third regularity result, which states that weak solutions belong to the domain of the low-dimensional second-order derivative operator L_μ (Chapter 6, Theorem 6.10).

The PhD thesis also includes relevant examples. In Chapter 4, the author illustrates the application of the existence and uniqueness theorem (Theorem 4.1) for parabolic equations on low-dimensional structures with Example 4.12. In Chapter 5, specifically Examples 5.11

and 5.12, the author demonstrates that elliptic equations on low-dimensional structures may have weak solutions that are not straightforwardly constructed by combining solutions from individual component manifolds of the structure, such as through gluing or summing operations.

In conclusion, the PhD thesis presented by M. Łukasz Chomienia constitutes a high-quality body of work, marked by the establishment of new results, the development of new concepts, and the generalization of methods in the field of partial differential equations within the framework of low-dimensional structures. For these reasons, I wholeheartedly support the PhD thesis defense of M. Łukasz Chomienia and recommend that the thesis be awarded with distinction.

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