Ph.D. Program in Mathematics

365 Fifth Avenue
New York, NY 10016-4309
212.817.8530
mathematics@gc.cuny.edu
http://www.gc.cuny.edu

THE GRADUATE CENTER
CITY UNIVERSITY OF NEW YORK

Report on *The Logical Strength of Ramsey-theoretic Principles Over a Weak Base Theory* by Katarzyna W. Kowalik

Katarzyna Kowalik's dissertation is a contribution to Reverse Mathematics. Reverse Mathematics started in the 1970s as a program proposed by Harvey Friedman and Stephen Simpson, with the aim to determine what axioms about infinity are necessary in proofs of theorems of mathematics. In order to introduce some terminology, let me start with a very brief description. The base theory in which all proofs about mutual dependencies between axiom systems and particular suitably formalized theorems is a weak fragment of second-order arithmetic, RCA_0, with the axioms declaring that all computable sets of natural numbers exist and that the numbers satisfy the induction schema for the restricted class of arithmetic Σ^0_1-formulas with second-order parameters, denoted by $\mathsf{I}\Sigma^0_1$. Using suitable coding, in RCA_0 one can formalize much of analysis, algebra, and more, but what can be proved is limited to what can be described as constructive mathematics. Over the years Reverse Mathematics grew to become an independent discipline in the foundations of mathematics. The original impact came from the discovery that many classical theorems of mathematics can be classified with respect to their proof-theoretic strength by showing that over RCA_0 each is equivalent to one of the five major fragments of second-order arithmetic, known as the big five. The three lowest levels being—in increasing order of strength—the theories RCA_0, WKL_0, and ACA_0; where WKL_0 is the extension of RCA_0 by the axiom expressing the Weak König's Lemma, which says that every infinite binary tree has an infinite path, and ACA_0 is the extension of RCA_0 obtained by adding the full König's Lemma for all infinite finitely branching trees.

For each $n \geq 3$, the Infinite Ramsey Theorem for n-tuple and k-colorings, RT^n_k, is equivalent over RCA_0 to ACA_0. The status of the Infinite Ramsey Theorem for pairs and 2-colorings is different. RT^2_2 is not provable in RCA_0 and $\mathsf{RCA}_0+\mathsf{RT}^2_2$ does not prove WKL_0. These and related results opened a whole area of investigations into reverse mathematics status of various infinitary combinatorial statements.

The $\mathsf{I}\Sigma^0_1$ axioms provide enough strength for the coding tasks needed to formalize large parts of mathematics in RCA_0, but it has been asked if there is a weaker set of number-theoretic axioms that would also be sufficient, and whether with such axioms the main results of Reverse Mathematics would stay unchanged. One candidate is the theory RCA^*_0, introduced by Simpson and Smith in 2009. In this theory, $\mathsf{I}\Sigma^0_1$ is replaced by the induction schema for formulas that are provably equivalent to formulas in which all quantifiers are bounded, with the additional axiom which declares that the exponential function is total. The theory WKL^*_0 is obtained from WKL_0 in the same way.

In her dissertation, Kowalik presents a large number of new results related to the areas of research that I outlined in the two preceding paragraphs. The thesis is devoted to the reverse mathematics status of RT^2_2 and related combinatorial statements, as well as to the impact on the main results of lowering the strength of the base theory from RCA_0 to RCA^*_0. The last chapter is devoted to proof size comparisons between selected axiom systems and their partially conservative extensions. The results in this chapter are obtained with the help of novel model-theoretic techniques developed in previous chapters.

Chapter 1 is a comprehensive and very well written introduction. The results and the techniques to prove them belong to the intersection of research on specific aspects of model theories of first-order and second-order arithmetics and their fragments, combined with nontrivial elements of infinitary combinatorics, computability theory, and proof theory. To collect all this material in one informative and very readable introduction is an accomplishment on its own.

Chapter 2 is devoted to a detailed analysis of the status of what is called *normal versions* of RT^n_k for $n, k \geq 2$. The shift from RCA_0 to RCA^*_0 brings in a whole new class of models for consideration, namely the models of RCA^*_0 in which $\mathsf{I}\Sigma^0_1$ fails. Such models always have Σ^0_1-definable proper cuts, and it follows that they also have unbounded definable sets—in the second-order part of the model—which can be enumerated in the increasing order by the elements of those cuts. This leads to two notions of infinite set. In the first, dubbed *normal*, a set is considered infinite if and only if it is unbounded, as it is a normal practice in the model theory of arithmetic. In the second *long* version, the elements of an infinite set are required to be in one-to-one correspondence with the domain of the model.

One of the crucial results is about a special class of second-order sentences called pseudo-second-order. It is a theorem that says that for every pseudo-second-order sentence χ and every model $(M, \mathcal{X})$ od RCA^*_0, if $I \in \mathcal{X}$ is a proper Σ^0_1-definable cut, then $(M, \mathcal{X}) \models \chi$ if and only if $(I, \mathrm{Cod}(M/I) \models \chi$, where $\mathrm{Cod}(M/I)$ is the set of all subsets of I that are coded in M. As a corollary one gets that if $\mathsf{I}\Sigma^0_1$ fails in $(M, \mathcal{X})$ then for every pseudo-second-order sentence χ, χ holds in $(M, \mathcal{X})$ if and only if it holds in $(M, \mathcal{Y})$, where $\mathcal{Y}$ is the set of Δ_1-definable subsets of M. Since the statements formalizing the normal versions of Ramsey's Theorem and some other Ramsey-like combinatorial facts that are considered in the thesis are equivalent over RCA^*_0 to pseudo-second-order sentences, the theorem and its corollary are applied to prove a number of results about conservativity of these principles over RCA^*_0, and to obtain axiomatizations of the first-order consequences of the theories $\mathsf{RCA}_0+\mathsf{RT}^n_2$, for $n \geq 3$.

In **Chapter 3** the attention shifts to the long versions of RT^n_k, the chain-antichain principle CAC, the ascending-descending sequence principle ADS, and the cohesive set principle COH, all denoted by adding ℓ in front of their normal names. The long ADS principle comes in two versions, $\ell\text{-}\mathsf{ADS}^{\mathrm{set}}$ and $\ell\text{-}\mathsf{ADS}^{\mathrm{seq}}$, depending on whether the solution set in the principle is required to be a set or a sequence, because it turns out that while RCA_0 proves that these two versions are equivalent, this is not the case for RCA^*_0. First, Kowalik proves that $\mathsf{RCA}^*_0+\ell\text{-}\mathsf{RT}^3_2$ is equivalent to ACA_0, showing how strong it is. The rest of the chapter is devoted to proving that the long versions of several other principles are logically much weaker. In particular she shows that over RCA^*_0 the long version of RT^2_2 and CAC imply their normal versions, and that CAC and $\ell\text{-}\mathsf{ADS}^{\mathrm{set}}$ imply $\ell\text{-}\mathsf{ADS}^{\mathrm{seq}}$, but neither of these implication reverses, and that $\ell\text{-}\mathsf{ADS}^{\mathrm{seq}}$ is equivalent to ADS. The rest of the chapter is devoted to the principle COH, which, because of its form, requires a separate treatment. It is known that COH is Π^1_1-conservative over RCA_0. In a recent paper, David Belanger proved some extensions of this result and asked whether COH is also Π^1_1-conservative over RCA^*_0. Kowalik gives a negative answer by showing how it follows from the results of Chapter 2 that COH is not Π_5-conservative over RCA^*_0.

The main result of the last section of Chapter 3 is that no model of RCA^*_0 of the form $(M, \mathcal{X})$, where $\mathcal{X}$ is the set of all Δ_1-definable subsets of M, satisfies COH, from which it follows that for any $n, k \geq 2$, $\mathsf{RCA}^*_0+\mathsf{RT}^n_k$ does not prove COH and that RT^n_k, CAC, and ADS are incompatible with COH with respect to implications over RCA^*_0.

Chapter 4, which takes good 36 pages, is devoted to a proof of a "non-speedup" theorem. While $\mathsf{RCA}^*_0+\mathsf{CAC}$ is $\forall\Pi^0_3$-conservative over RCA^*_0, Kowalik shows that adding CAC to RCA^*_0 does not result in essential shortening of proofs of $\forall\Pi^0_3$ statements that are provable in RCA^*_0. The proof is model-theoretic. It involves combination of various techniques including a version of forcing interpretations used to prove conservativity of extensions for fragments of second-order arithmetic, and restricted definable ultrapowers constructions that have been extensively used in the study of models of fragments of first-order arithmetic. The main result is that there is an algorithm that, for a given proof δ in WKL^*_0 of a $\forall\Pi^0_3$-sentence σ, produces a proof of σ in RCA^*_0 in time that is polynomial in the size of δ.

Assessment Katarzyna Kowalik's dissertation is an important contribution to the study of the logical strength of fragments of second-order arithmetic. As shown by the list of references, the work is in the area that has been intensively studied in recent years. The results are original and the techniques used in their proofs make this work interesting not just to the specialists in the field. All chapters are well written and they include careful and very readable introductions to several technically advanced topics.

Katrzyna Kowalik's dissertation amply satisfies all requirements for granting a PhD. In my opinion, it should be awarded an honorary distinction.

Roman Kossak

Roman Kossak

Professor Emeritus
rkossak@gc.cuny.edu

August 8, 2025

Appendix

Below is a list of my comments and suggestions for the author for future publications and presentations. They are mostly of editorial nature. Except for a few minor corrections, no revisions of the current text are necessary.

1. Page 19. I would add a definition of elementary computable functions, or a reference.
2. Page 24. "On the other hand it is known that $\mathsf{I}\Sigma^0_1$ proves Con(RCA^*_0)." I would add a reference.
3. Page 25. The Chong-Mourad lemma is stated without any comments. I would say a few words about what it is used for in the dissertation. .
4. Page 25. Bottom of the page. To avoid ambiguity, I would stress that Ramsey's theorem is about *ordered* n-tuples.
5. Page 33. It is not easy to parse Defintion 2.7. It would be helpful to give an example using one of the combinatorial principles first.
6. Page 35, line 10. strictly speaking $\mathcal{X} \cap \mathcal{P}(\tilde{H}), \ldots$ are not second-order structures.
7. Page 37, Corollary 2.13. It took me a while to realize that $\top$ is a not a T. Say: "constant symbol $\top$."
8. Page 43. Second paragraph of the proof of Theorem 2.23. Perhaps this corollary of Theorem 2.16 could be stated as a separate lemma.
9. Page 44, line 9. You say "the constructed model $M\ldots$", but M has not been introduced yet. This is done in the next paragraph. Announce earlier that you are constructing a model $M \models R^n + \varphi$.
10. Page 44, line -9. Theorem 1.21 is a collection of many result. Say which one you are using.
11. Page 45, line 9. "By [5]...". A more specific reference would be helpful.
12. Page 51, line -13. Notation $c \restriction [G_{i_j}]^2 = 0$ and $c \restriction [S]^2 = 1$ is a bit awkward.
13. Page 51, line -6. $\mathrm{I}\Delta_0 + \mathsf{superexp} \subsetneq \mathrm{I}\Sigma_1$. You use such notation here and in several other places in the thesis. It is clear what you mean, but theories are sets of sentences so this is not quite right.
14. Page 61. A short note about which fragment of PA proves Dilworth's theorem would be helpful here.
15. Page 63. I would add somewhere an explanation of the intended meaning of $s \vdash_\tau v \downarrow$.
16. Page 65. In (FI13), delete $\downarrow$.
17. Page 66. In Lemma 4.11, "...and the paragraph below it". Is this right? There is no paragraph "below it" in the version of [32] on arXiv. I did not check the published version. In any case, it would be helpful to quote that paragraph.
18. Page 73. In the definition of SC it seems that c depends on x. We seen in the proof on page 81 that is does not.
19. Page 76, line 13. What is "generic ultrapower"?
20. Page 77, line 3 of the proof of Lemma 4.25. What are the "three function symbols"?