UNIVERSITY OF LEEDS

REPORT ON PHD THESIS *THE LOGICAL STRENGTH OF RAMSEY-THEORETIC PRINCIPLES OVER A WEAK BASE THEORY* BY KATARZYNA KOWALIK

PAUL SHAFER

The logical strength of Ramsey-theoretic principles over a weak base theory by Katarzyna Kowalik examines the reverse mathematics of Ramsey's theorem for pairs and its consequences over the theory RCA_0^*.

Reverse mathematics is a longstanding program in mathematical logic at the intersection of proof theory and computability theory whose aim is to determine the precise axioms required to prove the classical textbook theorems of countable mathematics. It was introduced by Harvey Friedman and greatly expanded by Stephen Simpson [47] (in the author's references). The formal setting is second-order arithmetic, where the mathematical objects are natural numbers, sets of natural numbers, and anything that can be encoded thereby. The restriction to countable objects is not as severe as it might at first seem. Through coding and separability, one may discuss the real numbers, the basic topology of the real line, continuous functions on the reals, and, more generally, complete separable metric spaces and continuous functions between them. This thesis concerns countable combinatorial objects, and representing them in second-order arithmetic is straightforward.

The so-called *subsystems of second-order arithmetic* start with the basic axioms describing how the natural numbers behave with respect to $<$, $+$, and $\times$. One may take these to be the axioms of a discretely ordered commutative semi-ring with identity. Further strength is added along two dimensions: induction and comprehension. Induction axioms assert for which formulas φ the induction principle

$$\varphi(0) \wedge \forall n \big(\varphi(n) \to \varphi(n+1)\big) \ \to\ \forall n\, \varphi(n)$$

holds. For example, the Σ^0_1 *induction scheme* $\mathsf{I}\Sigma^0_1$ asserts that the above holds for all (universal closures of) Σ^0_1 formulas φ. The bounding axioms are closely related to the induction axioms. The bounding axiom for φ is

$$\forall n < a\, \exists m\, \varphi(n,m) \ \to\ \exists b\, \forall n < a\, \exists m < b\, \varphi(n,m),$$

and the Σ^0_1 *bounding scheme* $\mathsf{B}\Sigma^0_1$ (for example) asserts that the above holds for all Σ^0_1 formulas φ.

Comprehension principles assert which sets of natural numbers exist. The comprehension axiom for φ is the (universal closure of)

$$\exists X\, \forall n \big(n \in X \ \leftrightarrow\ \varphi(n)\big),$$

provided X is not free in φ. This says that one may form the set $X = \{n : \varphi(n)\}$. The Σ^0_1 *comprehension scheme* Σ^0_1-CA_0 (for example) asserts that the above holds for all Σ^0_1 formulas φ.

The usual starting point for reverse mathematics, i.e., the *base theory*, is the theory RCA_0, which consists of the basic axioms, the Σ^0_1 *induction scheme* $\mathsf{I}\Sigma^0_1$, and the Δ^0_1 *comprehension scheme* Δ^0_1-CA_0. RCA_0 stands for *recursive comprehension axiom* and roughly corresponds to computable

Date: August 15, 2025.

mathematics: if one wants to prove that a set exists, then one must give an algorithm for computing that set.

Large swaths of countable mathematics have been analyzed over RCA_0. This thesis concerns Ramsey's theorem and its consequences. The important principles are the following.

- Ramsey's theorem for n-tuples and k colors RT^n_k: For every $c\colon [\mathbb{N}]^n \to k$, there is an infinite $H \subseteq \mathbb{N}$ such that c is constant on $[H]^n$. Here $[\mathbb{N}]^n$ denotes the set of n-element subsets of $\mathbb{N}$ (encoded by increasing n-tuples), similarly for $[H]^n$, and k denotes the k-element set $\{0, 1, \dots, k-1\}$.

- The chain/antichain principle CAC: Every infinite partial order contains either an infinite chain or an infintie antichain.

- The ascending/descending sequence principle ADS: Every infinite linear order contains either an infinite ascending sequence or an infinite descending sequence.

- The cohesive principle COH: For every sequence of sets $(A_n : n \in \mathbb{N})$, there is an infinite set C such that $\forall n\,(C \subseteq^* A_n \vee C \subseteq \mathbb{N} \setminus A_n)$. Here $X \subseteq^* Y$ denotes *eventual containment*: that $X \setminus Y$ is finite.

Classic work of Jockusch [28] translated into the reverse mathematics framework shows that, for any fixed $n \geq 3$ and $k \geq 2$, RT^n_k is equivalent to the system ACA_0 over RCA_0. ACA_0 stands for *arithmetical comprehension axiom* and is the system obtained by adding the Σ^0_1 comprehension scheme to RCA_0. Thus ACA_0 proves RT^n_k and, so long as $n \geq 3$ and $k \geq 2$, $\mathsf{RCA}_0 + \mathsf{RT}^n_k$ proves all the axioms of ACA_0.

Of course ACA_0 also proves RT^2_2. That RT^2_2 is strictly weaker than ACA_0, i.e., that $\mathsf{RCA}_0 + \mathsf{RT}^2_2$ does not prove ACA_0, was a major breakthrough of Seetapun [46]. Furthermore, RT^2_2 is strictly stronger than CAC [24], which is strictly stronger than ADS [37]. Thus over the base system RCA_0, we have that

$$\mathsf{ACA}_0 \equiv \mathsf{RT}^n_k \text{ (for } n \geq 3,\ k \geq 2) \quad > \quad \mathsf{RT}^2_2 \quad > \quad \mathsf{CAC} \quad > \quad \mathsf{ADS},$$

where $\equiv$ means 'is equivalent to' and $>$ means 'is strictly stronger than' in terms of provability. Nowadays there is great interest in the first-order consequences of these principles. $\mathsf{RCA}_0 + \mathsf{RT}^n_k$ for $n \geq 3$ and $k \geq 2$ is equivalent to ACA_0, whose first-order consequences are exactly those provable in Peano arithmetic. On the other hand, the first-order consequences of $\mathsf{RCA}_0 + \mathsf{CAC}$ and $\mathsf{RCA}_0 + \mathsf{ADS}$ exactly those provable in $\mathsf{B}\Sigma_2$ [8]. The exact first-order consequences of $\mathsf{RCA}_0 + \mathsf{RT}^2_2$ are unknown and a major open question, though strong partial results have been obtained [35, 36].

In the base theory RCA_0, one typically justifies the Δ^0_1 comprehension scheme by saying that it represents computable definability and appealing to the fact that, classically, the Δ^0_1-definable sets are exactly the sets computable by Turing machines. The choice of starting with the Σ^0_1 induction scheme in RCA_0 is perhaps more arbitrary. A popular mathematical reason for choosing $\mathsf{I}\Sigma^0_1$ is that it provides a robust notion of a subset of $\mathbb{N}$ being infinite that is lost when moving to weaker induction schemes. How RT^n_k, CAC, and ADS relate to each other when $\mathsf{I}\Sigma^0_1$ is lacking is exactly the topic of this thesis.

The author works over the theory RCA^*_0, which is the weakening of RCA_0 obtained by replacing $\mathsf{I}\Sigma^0_1$ by $\mathsf{B}\Sigma^0_1$ plus the assertion that the exponential function is total. ($\mathsf{B}\Sigma^0_1$ does not suffice to prove that the exponential function is total, so that assumption is added as an axiom in order to make finite sets and sequences easy to encode.) When $\mathsf{I}\Sigma^0_1$ is lacking, there is a distinction between a subset of $\mathbb{N}$ being unbounded and being bijective with $\mathbb{N}$. Over RCA^*_0, a set that is bijective with $\mathbb{N}$ is unbounded, but the converse may not hold. Thus one must say exactly what one means when asserting that a set is 'infinite,' such as in the conclusion of Ramsey's theorem and its consequences. The versions of RT^n_k, CAC, and ADS where 'infinite' is interpreted as 'unbounded' are called the *normal* versions of these principles and are studied in Chapter 2.

One of the main technical tools in Chapter 2 is that if $\mathsf{I}\Sigma^0_1$ fails, then the Ramsey-theoretic principles are first-order principles in the sense of having equivalent *pseudo-second-order* formulations that hold if and only if they hold on Σ^0_1 proper cuts (Theorem 2.9). This leads to a slick conservativity criterion for pseudo-second-order statements over RCA^*_0 (Theorem 2.12). The following are equivalent for a pseudo-second-order sentence ψ.

- $\mathsf{RCA}^*_0 + \psi$ is Π^1_1-conservative over RCA^*_0 (i.e., the two theories prove the same Π^1_1 sentences).
- $\mathsf{RCA}^*_0 + \neg\mathsf{I}\Sigma^0_1 \vdash \psi$.
- $\mathsf{WKL}^*_0 \vdash \psi$. (Here WKL^*_0 is RCA^*_0 plus *weak König's lemma*: every infinite binary tree has an infinite path.)

This in turn allows the author to separate RT^3_2, RT^2_2, CAC, and ADS over $\mathsf{RCA}^*_0 + \neg\mathsf{I}\Sigma^0_1$ and also by their first-order parts over RCA^*_0 (Corollary 2.13). Let P and Q be principles from the list above with P to the left of Q. Then both $\mathsf{RCA}^*_0 + \neg\mathsf{I}\Sigma^0_1 + \mathsf{Q} \nvdash \mathsf{P}$ and there is an arithmetical sentence provable in $\mathsf{RCA}^*_0 + \mathsf{P}$ but not in $\mathsf{RCA}^*_0 + \mathsf{Q}$. This contrasts with the situation over RCA_0 where, for example, $\mathsf{RCA}_0 + \mathsf{CAC}$ and $\mathsf{RCA}_0 + \mathsf{ADS}$ have the same first-order parts. Regarding conservativity, the author shows that the Ramsey-theoretic principles mentioned here are all $\forall\Pi^0_3$-conservative but not Π_4-conservative over RCA^*_0 (Theorem 2.17). Finally in Chapter 2, the author exactly characterizes the first-order consequences of $\mathsf{RCA}^*_0 + \mathsf{RT}^n_2$ for each fixed $n \geq 3$ (Theorem 2.21). Let R^n be the first-order theory axiomatized by $\mathsf{B}\Sigma_1$, the totality of the exponential function, and the sentences

$$\mathsf{B}\Sigma_\ell \;\rightarrow\; (\mathsf{B}\Sigma_{\ell+1} \vee \Delta_\ell\text{-}\mathsf{RT}^n_2)$$

for $\ell \geq 1$. Here Δ_ℓ-RT^n_2 is a Δ_ℓ definable version of RT^n_2, see Definition 2.11. Then R^n axiomatizes the first-order consequences of $\mathsf{RCA}^*_0 + \mathsf{RT}^n_2$. It follows that the first-order parts of $\mathsf{RCA}^*_0 + \mathsf{RT}^n_2$ for $n \geq 3$ are all different and not finitely axiomatizable (Theorem 2.23). Contrast this with the situation over RCA_0, where the first-order part of $\mathsf{RCA}_0 + \mathsf{RT}^n_k$ for each $n \geq 3$ corresponds to Peano arithmetic.

Chapter 3 is a short chapter analyzing the *long* versions of the Ramsey-theoretic principles, where 'infinite' is now interpreted as 'bijective with $\mathbb{N}$.' With ADS, there is now a distinction between whether the ascending/descending sequence in the conclusion is rendered as a set ($\mathsf{ADS}^{\mathsf{set}}$) or as a sequence ($\mathsf{ADS}^{\mathsf{seq}}$). Theorem 3.2 shows that the long versions of RT^3_2, RT^2_2, CAC, and $\mathsf{ADS}^{\mathsf{set}}$ imply $\mathsf{I}\Sigma^0_1$ and hence RCA_0 over RCA^*_0. Thus the long versions of these principles take us back to their classical analyses over RCA_0. Interestingly, the long version of $\mathsf{ADS}^{\mathsf{seq}}$ is equivalent to the normal version of ADS over RCA^*_0 (Theorem 3.7). Chapter 3 also analyzes COH over RCA^*_0. The main results are that:

- $\mathsf{RCA}^*_0 + \mathsf{COH}$ is not Π_5-conservative over RCA^*_0 (Corollary 3.12);
- RT^n_k for $n, k \geq 2$, CAC, and ADS are all incomparable with COH over RCA^*_0 (Corollary 3.17).

Again, this contrasts with the situation over RCA_0. COH is Π^1_1-conservative over RCA_0 [5] and is implied by ADS (and hence all the other principles mentioned here) over RCA_0 [24].

Chapter 4 is dedicated to proving that $\mathsf{WKL}^*_0 + \mathsf{CAC}$ is polynomially simulated by RCA^*_0 with respect to $\forall\Pi^0_3$ sentences (Theorem 4.6). This means that there is a polynomial-time algorithm that, given a proof of such a sentence in $\mathsf{WKL}^*_0 + \mathsf{CAC}$, outputs a proof of the same sentence in RCA^*_0. This can be seen as a strong conservativity of $\mathsf{WKL}^*_0 + \mathsf{CAC}$ over RCA^*_0 for $\forall\Pi^0_3$ sentences. Not only do the two theories prove the same such sentences, but they prove the same such sentences with roughly the same efficiency. Contrast this with the situation for RT^2_2: again for $\forall\Pi^0_3$ sentences, $\mathsf{RCA}_0 + \mathsf{RT}^2_2$ is polynomially simulated by RCA_0, but $\mathsf{RCA}^*_0 + \mathsf{RT}^2_2$ has super-exponential speedup over RCA_0^* [32]. The proof of the non-speedup result for $\mathsf{WKL}_0^* + \mathsf{CAC}$ is elaborate and works roughly as follows. First, reduce to the difficult case of interpreting $\mathsf{WKL}_0^* + \mathsf{CAC}$ in the theory $\mathsf{RCA}_0^* + \neg\mathsf{I}\Sigma^0_1 + \Sigma^0_1\text{-}\mathsf{LPC}$, where Σ^0_1-LPC says that the cut I^0_1 is Σ^0_1-definable. Then compose two forcing interpretations. The first interprets the theory $\mathsf{I}\Delta_0 + \mathsf{exp} + \mathsf{SC}$ in $\mathsf{RCA}_0^* + \neg\mathsf{I}\Sigma^0_1 + \Sigma^0_1\text{-}\mathsf{LPC}$ by simulating an effective ultrapower construction. Here exp asserts the totality of the exponential function, and SC is a technical axiom guaranteeing that a certain cut is short in a certain sense. The second interprets $\mathsf{WKL}_0^* + \mathsf{CAC}$ in $\mathsf{RCA}_0^* + \neg\mathsf{I}\Sigma^0_1 + \Sigma^0_1\text{-}\mathsf{LPC}$ by mimicking the $\forall\Pi^0_3$-conservativity proof of $\mathsf{RCA}_0^* + \mathsf{CAC}$ over RCA_0^*.

This is an excellent thesis that easily satisfies all requirements for a PhD in mathematics. Reverse mathematics is a delicate subject to begin with, and reverse mathematics over the extra-weak base theory RCA_0^* is even more so. The thesis shows mastery of diverse subjects such as combinatorics related to the infinite version of Ramsey's theorem, the model theory of arithmetic with limited induction, forcing interpretations, and more. The thesis is also exceptionally well-written. All-in-all, it is a remarkable piece of work. Kowalik has already published articles [15, 31] on the material from Chapters 2 and 3. These articles appear in *Annals of Pure and Applied Logic* and *Journal of Symbolic Logic*, both of which are excellent mathematical logic journals. Kowalik is also preparing [34] based on the impressive material from Chapter 4. The resulting article should be publishable in a top mathematical logic journal or perhaps even a generalist journal such as *Transactions of the American Mathematical Society*. Kowalik is already a valued member of the mathematical logic community. She has participated in many conferences, workshops, and research visits, and she gives excellent talks. She is perfectly poised to continue in mathematical research.

Recommendation

To be clear, I deem the thesis *The logical strength of Ramsey-theoretic principles over a weak base theory* by Katarzyna Kowalik to be sufficient to grant a PhD. Furthermore, I move that the thesis be awarded an honorary distinction.

The comments on the following pages are entirely optional and are included primarily for the author's interest.

Signed
Paul Shafer
School of Mathematics
University of Leeds

The following is a list of minor suggestions and curiosities. Nothing is essential. PX LY denotes page X, line Y.

- It is nice that there are a number of open questions raised throughout the thesis. It would be even nicer to add another section (at the very end, say) that lists the questions again all in one place for easy reference.

P9: Are there other types of infinity besides *normal* and *long* when working in RCA_0^*? For example, could something like "if $\mathsf{I}\Sigma_1^0$ fails, then longer than I_1^0" make sense?

P13 L35: It is implicit, but still it would be good to write that the M in a $\mathcal{L}_{\mathrm{II}}$-structure $(M, \mathcal{X})$ is an $\mathcal{L}_{\mathrm{I}}$-structure.

P13 L40: It would be better to explain that 'ω-model' can either mean "has ω as the first-order part" or "has the same first-order part as the model currently under discussion," but that the meaning is always clear from context.

P14 L4: The description of PA^- might also need to include that there is a multiplicative identity.

P14 L18: The parameters in $\Gamma\text{-}\mathrm{Def}(M)$ can be both first- and second-order parameters (from M) right?

P14: For the second-order version of (1.2), what happens if the universal closure in $\mathsf{I}\Delta_n^0$ is changed to the following (where φ is Σ_n^0 and ψ is Π_n^0)?

$$\forall X \forall n\big(\varphi(X,n) \Leftrightarrow \psi(X,n)\big) \;\Rightarrow\; \forall X\big(\varphi(X,0) \wedge \forall n(\varphi(X,n) \Rightarrow \varphi(X,n+1)) \;\Rightarrow\; \forall n \varphi(X,n)\big)$$

That is, what happens if the equivalence of φ and ψ needs to hold for all X before induction on φ (with any parameter X) is allowed?

P15 L31: What does "proves the recursive equations implicitly defining them" mean?

P15 L32: Please give a reference for the fact that $\mathsf{I}\Sigma_1$ is the weakest system in the hierarchy proving exp and supexp.

P16 L22: The phrase 'universal formulas' is confusing here.

P17: In (1.9), X should not appear freely in φ.

P19: Please give a reference for the provably total functions of RCA_0 and of RCA_0^*. Please also give a reference for the first-order parts of RCA_0^*, RCA_0, and ACA_0.

P19: How does RCA_0^* compare to the system PRA^2 (second-order primitive recursive arithmetic) from Bazhenov, Fiori-Carones, Liu, Melnikov, *Primitive recursive reverse mathematics*, Annals of Pure and Applied Logic, Volume 175, Issue 1, Part A, 2024?

P20: Does Theorem 1.6 also hold for extensions of RCA_0^*?

P21 L3: X' is an unfortunate choice of notation in a document that also discusses Turing jumps.

P21: In Theorem 1.10, WKL means just the statement that infinite binary trees have paths, right? Not a system with full induction?

P24 L11: Please explain what formula classes like $\forall\Pi^0_3$ are.

P24: What is X in the definition of I^0_1 (1.13)? Maybe it would be better to write that "every unbounded set has a finite subset of cardinality x" without giving the unbounded set a name that is not used.

P25: Regarding (1.14), Ramsey-theoretic statements also usually have the property that if Y is infinite, $\Psi(X,Y)$ holds, and $Z \subseteq Y$ is infinite, then $\Psi(X,Z)$ also holds.

P27 L3: Usually that C must be infinite/unbounded is part of the definition of C being cohesive.

P27: Is there such a thing as a COH principle but only for short sequences of sets $(R_i)_{i\in I}$?

P31 L9: It would be good to point out somewhere why such definitions are Δ^0_1. (Because every x is in exactly one such interval.)

P31: Does RT^3_2 still imply ACA_0 over RCA^*_0?

P33: Why are not the results of [15] included in the thesis?

P34 L14: The symbol I appears twice at the beginning of this line.

P34: Since this is a thesis, it would be better to include the full proof of Theorem 2.9, not just the proof for a unary relation symbol.

P35 L5: Please clarify in what system the argument that $(H,\leq,R)$ satisfies γ is taking place.

P35 L30: This paragraph is difficult to follow. What would it mean for the equivalence to depend on the cut?

P35: In Corollary 2.10, are the sets of Δ_1- $\mathrm{Def}(M,A)$ allowed to be definable with parameters? Both first- and second-order?

P36: It would be good to give an example of a σ and the translated Δ_ℓ-σ.

P36: Is there also a reference for the equivalence claimed in (2.1)?

P37: Are there proofs of Corollary 2.13 that do not rely on difficult ω-model separations?

P39 L37: It would be good to include a precise statement of the relativized limit lemma being used here.

P40 L18 & L26: That picking the 'least such x_1, x_2' (L18) and the 'smallest z' (L26) uses the appropriate amount of induction should be mentioned.

P41 L35: This induction is happening externally, right?

P42: Regarding Theorem 2.21, what can be said about the Π^1_1-consequences of $\mathsf{RCA}_0^* + \mathsf{RT}^n_2$?

P44 L31: Theorem 1.21 does not seem relevant here. Was a different theorem meant?

P46: Please also remind the reader what 'of cardinality $\mathbb{N}$' means because this is the start of a new chapter.

P46: What happens for unbalanced versions of these principles? Such as:

- Every infinite linear order has either an unbounded ascending sequence or a long descending sequence.
- Every infinite partial order has either an unbounded chain or a long antichain.
- Every infinite partial order has either an unbounded antichain or a long chain.
- Every coloring $c\colon [\mathbb{N}]^2 \to \{0,1\}$ has either an unbounded homogeneous set for color 0 or a long homogeneous set for color 1.
- etc.

P49: What can be said about the conservativity of GGP^2_2 over $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$?

P50 L12: This paragraph is too vague and needs more detail.

P53: In the Σ^0_2-separation principle and similar, the solution Δ^0_2 set is also allowed to be defined from parameters, right?

P64 L1: In notation like $\varphi(\bar{v}, \bar{u}) \Rightarrow \psi(\bar{v}, \bar{w})$, is it assumed that $\bar{v}$, $\bar{u}$, and $\bar{w}$ are all distinct lists of variables? In particular, that there is no overlap between $\bar{u}$ and $\bar{w}$?

P66 L3: This paragraph mentions suppressing the τ subscripts but then writes them anyway.

P68 L4 & L5: In displayed items (a) and (b), should the $\bar{v}$'s be names? I.e., $\forall \bar{v} \in \mathrm{Name}_\tau$. Similarly in several other places.

P68: Please also give a reference for Lemma 4.16.

P69: Please add some more delimiters or extra space in the displayed formula (4.9) to make it easier to read. The formulas around the '$\wedge$' symbol run together.

P73: Please rephrase the SC statement at the top of the page so that the correct order of the quantifiers is obvious.

P76: Could a Π^0_1 cohesive set (i.e., the complement of a maximal r.e. set) be useful for the computable ultrapower discussion? Do maximal r.e. sets exist without $\mathsf{I}\Sigma^0_1$?

P76 L11: Is $\mathcal{X}$ still assumed to be countable here?

P76: Is the forcing translation presented in Definition 4.24 original? Please clarify and give a reference if appropriate.

P77 L5: Why not just write $\mathbb{N}$ instead of $\{x \in \mathbb{N} : x = x\}$?

P82 L5: Item (v) is a little mysterious at first. The explanation appears later on, but it would be good to include some motivation here or immediately following Definition 4.28.

P82 L30: There should be another '$<$' before $s_{\lfloor \frac{c}{2} \rfloor}$.

P84: Should the τ's be τ_2's in the displayed formula (4.19) at the top of the page?

P85 L25: Where in this argument does it matter that the quantifiers are bounded?

P87 L30: Does "$\preceq$ is a partial order on $s_1 \times s_1$" mean that $\preceq \cap (s_1 \times s_1)$ is a partial order? If not, what does it mean?

P88 L20: The $\varphi(x)$ in $\exists X \exists x\, \varphi(x)$ should probably be $\varphi(X, x)$.

P88 L32: It might be worth adjusting the variable names in $\theta^*(X, x, y, z, z')$ because it looks like the z' in θ^* plays the role of z in θ.

School of Mathematics, University of Leeds, Leeds, LS2 9JT, United Kingdom

Email address: `p.e.shafer@leeds.ac.uk`

URL: `https://peshafer.github.io`