REPORT FOR THESIS OF JOACHIM JELISIEJEW

DIANE MACLAGAN

The thesis *Hilbert schemes of points and their applications*, by Joachim Jelisiejew, definitely meets the standards for the award of a PhD. **I strongly recommend that the degree be awarded**.

The main topic of this thesis is the Hilbert scheme of points, and its Gorenstein locus. This is a subject with a rich history in algebraic geometry going back fifty years, but for which there are still many surprisingly elementary open questions.

The Hilbert scheme, introduced by Grothendieck, is one of the fundamental moduli spaces in algebraic geometry. It parameterizes subschemes of an ambient quasiprojective variety X. The Hilbert scheme of points is one of the more studied Hilbert schemes, and in special cases has much more structure. However, as highlighted in this thesis, it is not even known when these are irreducible (Problems 1.17 and 1.18).

While early work on the Hilbert scheme of points, such as Fogarty's result that $\mathrm{Hilb}^r(X)$ is smooth and irreducible when X is a smooth surface, and Nakajima's connection to Heisenberg algebras, was core algebraic geometry, there has also been a strong thread of work on this subject phrased almost entirely in commutative algebraic language. There has been little connection between these communities, and little overlap in results. **A major strength of this thesis is that Jelisiejew uses techniques and ideas from both communities.**

The main content of the thesis is in chapters 5, 6, and 7. The earlier chapters mostly review background on the commutative algebra approach to the Hilbert scheme of points, most notably Macaulay's inverse systems, and the basics of the Hilbert scheme of points, though there is some new material towards the end of chapters 3 and 4.

The main result of Chapter 5 is Theorem 5.1, which shows the equivalence of several different notions of smoothing a finite subscheme of a smooth variety X. In particular, a point of $\mathrm{Hilb}^r(X)$ lies in the "main" component consisting of the closure of the locus of r distinct reduced points if and only if it is "abstractly smoothable" (meaning that it is a special fiber of a flat family over some base for which the general fiber is smooth). This is done carefully to have no dependence on the field. The second half of the chapter consists of examples of nonsmoothable schemes. These are divided in cases that live in families that are either smaller, or larger, than the main component. The main contribution here is cataloguing the extent of our ignorance on this topic. The section finishes with the result that the family of "very compressed" ideals is smoothable for at most 95 points in $\mathbb{A}^3$. This (sadly!) shows that one of Iarrobino's methods to construct reducible Hilbert schemes of points in $\mathbb{A}^3$ cannot be improved.

The main result of Chapter 6 is a study of the Gorenstein locus of the Hilbert scheme of at most 14 points. For at most 13 points this locus is shown to be irreducible. For 14 points there are nonsmoothable finite schemes. These can all be embedded into $\mathbb{A}^6$, and the second result of the chapter is an explicit description of the Gorenstein locus of $\mathrm{Hilb}_1 4(\mathbb{A}^6)$. This is shown to be the union of two components, the second of which corresponds to ideals with Hilbert function $(1, 6, 6, 1)$. Jelisiejew gives an explicit description of this second component, and of the intersection of the two components. The technique to show irreducibility is a clever use of induction. By the previous chapter if irreducible components of a finite scheme can be smoothed, the whole scheme can be. Thus for the smallest possibly reducible case all reducible schemes, and all limits of reducible schemes, lie in the smoothable component. The main idea is then a clever deformation of an arbitrary scheme to a reducible one.

The main result of Chapter 7 is that the Gorenstein locus of the punctual Hilbert scheme has the expected dimension for at most 9 points. The proof technique is to compare the dimension of any family of such ideals to that of the curvilinear locus (called "aligned" in this thesis), which is the expected dimension. A key reduction step is to reduce to the case that the embedding dimension of the ideals in a family is maximal, and then classify the possibilities. Examples are given to show that the curvilinear locus is not the whole of this Hilbert scheme in general.

These are all interesting, solid contributions to the literature, and together far more than suffice for the award of a PhD.

The thesis is on the whole well written (particularly judging by the standards of theses).

The introduction is a particularly nice summary of the state-of-the-art in the subject, summarizing applications, history, and open problems. This is the best-written part of the thesis, and deserves to be more widely read.

At a local level, the only typos I noticed were all ones that would not be caught by a spell-checker.

At a global level, the decision to split the thesis into the algebraic background and the geometric applications made the thesis significantly harder to read, as the earlier sections were essentially unmotivated. As noted above, one of the strengths of Jelisiejew's work is that he crosses the divide between more algebro-geometric and more commutative algebraic approaches to these topics. For most of the problems considered here the algebraic geometry is the motivation, so having the algebra part earlier and disconnected made it fairly unmotivated. Inside individual chapters the writing would have benefited from more local motivation of individual definitions and lemmas, and explanation of the overall logic of the chapters.

I have the impression that some of these problems were caused by extracting "Jelisiejew's contribution" from joint works. This often stripped out the motivation for the problems. If it is a requirement of the University of Warsaw that only sole-authored work be included I would strongly recommend a review; this is not the way mathematics is done in the 21st century. Indeed, a strength of Jelisiejew's published work is that he has applied technical work on the Hilbert scheme of points in areas where the application was not obvious, but this is only hinted at in the thesis.

In addition, the exposition could be improved in the following places, where I was not able to fill in the gaps myself.

(1) p58 Proof of Proposition 4.38. Why are the curves C_0 and C_1 connected? For C_0 this is presumably by a Bertini variant; what hypotheses are in effect for the field?
(2) p64 Proof of Proposition 4.59. Why does it suffice to show this?
(3) p64 Example 4.61. What if $\mathbf{w} = 0$, so does not span a line?
(4) p79 Remark 5.32. In the previous part of the section you only use that $\mathrm{Hilb}_r^{sm}(\mathbb{P}^n)$ is rationally chain connected. This is true for any Hilbert scheme by Hartshorne's connectedness proof.

I finish this report with a list of typos. These are almost all very minor.

(1) p10 line 12. "Later Eisenbud-Buchsbaum result". Missing "the". This problem occurs multiple times later in the thesis.
(2) p16 Proof of Lemma 2.5. "then every its localisation". Delete "its".
(3) p16. Proposition 2.9(4). A should be ω_A.
(4) p26 line -3. "manned" should be "manner".
(5) p28 line 2. Should I_p be $P_{\geq p}$?
(6) p28 Definition 3.12. "acts a" should be "acts as".
(7) p31 Proof of Lemma 3.2.3, line 3. "$= 0$" missing at start of line.

(8) p34 The sentence including equation 3.38 is a run-on sentence (the comma should be a different punctuation mark; a period is probably best). This problem occurs multiple other times in the thesis.
(9) p37 In the paragraph after definition 3.49, the sentence "The polynomial f ..." is ungrammatical and unclear.
(10) p41 Example 3.61. First sentence "Assume that ...". "be" should be "is".
(11) p48 line 2. "others" should be "other".
(12) p50 line 3. Reference to Figure 4.2 should be to 4.1. This is not the only similarly wrong reference in the thesis.
(13) p52 line 1 aX should be X. Line 6. "We also showed" should be "He also showed". The next sentence seems to wrongly imply that $\mathrm{Hilb}^r(\mathbb{A}^n)$ did not exist before 2007. In the next paragraph, "or degree r" should be "of degree r".
(14) p53 Paragraph after Example 4.13. No comma before "which". This occurs several times.
(15) p54 Last paragraph of section 4.1. "is translates by" should be "is translated by". In the next sentence, is "unity" meant to be "identity"?
(16) p56 example 4.24. "arbitrary" should be "arbitrarily".
(17) p57 line 6. "each its" should be "each".
(18) p60 paragraph after first displayed equation. "is some subspace of functional". "functional" should be "functionals". Reference to section 3.3 should be to section 3.1.
(19) p62 End of proof of Proposition 4.50. "assess" is the wrong word here.
(20) p72 Proof of Lemma 5.9. Line 4. Do you mean "Let $\mathfrak{p} \subset A$ be any prime ideal mapping ..."?
(21) p73 Example 5.10. What do you mean by an $\mathbb{R}$-point of $\mathbb{C}[[x]]$?
(22) p74 Proof of Lemma 5.14. By an open affine neighbourhood $U \subset X$ of $f_t(p)$ do you mean an open affine neighbourhood $U \times T$ of $f(p)$?
(23) p75 Proof of Theorem 5.1. "A" missing before "smooth variety".
(24) p86 Equation (5.61). Should the subscript be on $\mathcal{B}_s$ here?
(25) p88 line 1. "we" missing before "prove".
(26) p92 Proof of Corollary 6.11. "let" missing before $\alpha_1^\nu - \delta$.
(27) p93 line 1. "an finite".
(28) p93 Above Definition 6.16. Should S be $\hat{S}$ here?
(29) p116 line -3. "devices"
(30) p118 line -5. Should Example 6.42 be 7.16 or 7.17?

(31) p119 paragraph above Lemma 7.12. The sentence beginning “In particular” seems to be missing something.
(32) p120 Second paragraph of the proof of Lemma 7.13. “less that” should be “less than”.
(33) p121 line 1. “constrains” should be “constraints”.