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# Report on PhD thesis of Daniel Wysocki "Geometric aproaches to Lie bialgebras, their classification and aplications"

## 1 The subject and results of the thesis

The main results of this thesis concern with the classification of $r$-matrices on low dimensional, $d = 3$ or $d = 4$, Lie algebras. Let us remind that $r$-matrices are solutions to the classical Yang-Baxter equations, and as such are related to exactly solvable models in both quantum mechanics (C.N. Yang) and statistical physics (R. J. Baxter). In mathematics, the $r$-matrices are important as they equip a Lie algebra with a structure of a Lie bi-algebra, which is an infinitesimal version of a Poisson-Lie group. Such structures are very heavily used in the integrable systems theory.

The two central results of the thesis are included in its Chapters 2 and 3. These consist in a classification and a construction of all nonequivalent $r$-matrices on all 3-dimensional real Lie algebras, and in a classification and a construction of all nonequivalent $r$-matrices on all *indecomposable* 4-dimensional real Lie algebras.

In both cases, the 3-dimensional and the 4-dimensional one, new methods are introduced to obtain the classifications.

The classification of $r$-matrices on 3-dimensional Lie algebras was known before, but the thesis employes new algebraic tools based on a certain gradation in the Grassman algebra, which enables Daniel Wysocki to reproduce the results of the known classification from scratch.

The *full* classification of nonequivalent $r$-matrices on 4-dimensional Lie algebras is unknow up to now, but the thesis introduces a totally new geometric approach, based on the Daniel Wysocki and his collaborators' concept of Darboux families, which is skilfully used to obtain the full classification of $r$-matrices on indecomposable 4-dimensional Lie algebras.

In addition to these two main results Chapter 2 goes beyond the 3-dimensional classification and shows how the algebraic method discussed in this Chapter can be used in dimension 4, to calculate $r$-matrices on a specific 4-dimensional Lie algebra $\mathfrak{gl}_2$. Since this algebra is decomposable, it is also used in Chapter 3 to show that the thesis-developed geometric method based on Darboux families enables to find $r$-matrices also on decomposable 4-dimensional Lie algebras, such as $\mathfrak{gl}_2$. Actually the method of obtaining $r$-matrices described in this section will work for any dimension grater than or equal to 4.

Chapter 4 of the thesis gives possible applications of the results of Chapter 3. In particular it shows how the $r$-matrices constructed in Chapter 3 can be used to construct and to study Hamiltonian systems associated with the Poisson structures corresponding to these

$r$-matrices. This Chapter introduces also a new notion of the foliated automorphic Lie systems, which enable for a generalization of Ermakov systems.

All the new results from Chapter 2-4 were published in respectable mathematics and mathematical physics journals (J. of Lie Theory, Symmetry, J. Phys. A). Chapter 5 includes unpublished material. It concerns with an extension of previously known deformations of Lie systems to systems which require a Jacobi or contact setting description. It provides examples of such extensions.

The thesis has a very nice introductory part in Sections 0 and 1. As the whole thesis, this part is well written, introduces reader into the subject (Chapter 1) and includes a summary of the mathematics needed to understand its main parts.

## 2 Evaluation and conclusion

I consider this thesis as very good: It provides new results, introduces new techniques and is in the subject that is in the heart of exact solutions part of very important fields of mathematical sciences encircled by such areas as integrable systems, quantum mechanics and statistical physics. The author of the thesis has proven that he is a very creative mathematician. He is also a very good writer.

In conclusion I state that the thesis fulfills all the requirements needed for a PhD thesis in the field of Mathematics, and that in my opinion Mr Daniel Wysocki deserves a PhD in Mathematics with distinction ('cum laude').

Paweł Nurowski

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