

A referee report on Doctoral Thesis of Daniel Wysocki “Geometric approaches to Lie bialgebras, their classification, and applications”

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1 Justification of the topics of the Thesis

The slogan “Yang–Baxter equation” is probably one of the most popular phrases in theoretical and mathematical physics literature in the last five decades (Google exposes about 17 800 000 results related to this phrase). Since its appearance in the works of C. N. Yang [185,186]¹ and R. J. Baxter [31] on exactly solvable models in quantum and statistical mechanics at the turn of the sixties and seventies of the 20-th century the equation named after these two authors became one of the central topics for investigations due to its great importance in the field of quantum and classical integrable systems. In the beginning of the eighties due to the efforts of Faddeev, Sklyanin, Drinfeld and many others this equation was put into the foundations of mathematical theories of Poisson–Lie groups and then quantum groups. In finite-dimensional and classical (as opposite to quantum) context the Yang–Baxter equation (CYBE) is an algebraic equation on an element $r \in \bigwedge^2 \mathfrak{g}$ of the second exterior power of a Lie algebra \mathfrak{g} which provides the appearance of the so-called Lie bialgebra structure on \mathfrak{g} , the infinitesimal version of a Poisson–Lie group, i. e. a Lie group with a Poisson tensor compatible with the multiplication. Besides, the infinite-dimensional CYBEs were studied intensively. Their solutions, which this time are elements of the second exterior power of the infinite-dimensional “loop Lie algebra” $\mathfrak{g}[\lambda^{-1}, \lambda]$ of Laurent polynomials or power series of the parameter λ with values in finite-dimensional \mathfrak{g} , give rise to infinite dimensional Lie bialgebras (or double Lie algebras in the approach of Semenov-Tian-Shansky [162]), which are a very effective tool in constructing and study of finite- and infinite-dimensional integrable systems.

It is worth mentioning that in the first decade of our century the theory of CYBE was developed in new direction by Polishchuk and others [Pol02, Pol09], who introduced and partially classified the solutions of the so-called associative Yang–Baxter equation (AYBE). Roughly speaking in this theory the Lie algebra structure is substituted by an associative one, which allows to produce new solutions even in the case of $\mathfrak{g} = \mathfrak{gl}(n)$ i.e., when the Lie algebra structure is the antisymmetrization of an associative product. The theory of AYBE also has many applications in integrable systems (see for instance [KZ19]).

From the beginning the problem of classification of the solutions of CYBE, called (classical) r -matrices, was considered and attempts to solve it were made by many authors, among which Belavin, Drinfeld and Stolin probably should be mentioned first of all [35,36,169]. This classification mainly

¹The bibliographical references numerated by numbers refer to the list of references of the Thesis. Additional references will be presented in the amsalpt: a bibliography style.

concerned semi-simple Lie algebras \mathfrak{g} and their loop algebras, however the problem of classification for general Lie algebras was treated as well and is still very important.

The results of the thesis under consideration (Thesis hereafter) can be divided into two parts (Chapters 2-3 and 4-5 correspondingly), first of which (the second one will be discussed below) is devoted to classification matters of r -matrices on low dimensional Lie algebras. The classification is obtained for 3-dimensional real Lie algebras and 4-dimensional indecomposable real Lie algebras. In the former case the results agree with the known ones [84,95], however they are obtained by new methods (described below). Taking into account the above mentioned discussion the topic of the first part of the Thesis appears to be very important from the mathematical point of view as well as from the point of view of possible physical applications.

The results in the second part of the Thesis mainly concern Lie systems and their generalizations. A Lie system is a system of differential equations on a smooth manifold, given by a nonautonomous vector field X , admitting a superposition rule with the help of which one can express the general solution to the system. In a sense such systems are similar to that integrable in the sense of Liouville: they are rare among general differential equations but quite important. Study of such systems goes back to S. Lie and G. Scheffers [116] and is related to finite-dimensional algebras of vector fields on a manifold. The interest to such systems was renewed in the beginning of this century (mainly by J. F. Cariñena and his collaborators one of which is J. de Lucas, the advisor of this Thesis) and is supported up to now due to their multiple applications to ordinary and partial differential equations [13–15, 46–56, 100]. A generalization of Lie systems called foliated Lie systems is introduced in the second part of the Thesis and studied thoroughly. As an application a new class of generalized Ermakov systems is established and studied as well as a class of systems related to r -matrices.

Finally in the last chapter of the Thesis a specific deformations of Lie systems are constructed with the help of Jacobi structures. In particular, examples related to r -matrices are considered.

Summarizing, the results of the second part of the Thesis fit and extend the above mentioned series of modern results on Lie systems, hence are important and up-to-date.

2 Description of the contents of the Thesis and the methods of investigation

The Thesis consists of Introduction, six chapters, two appendices, and list of references. Introduction contains a historical overview of results concerning Yang–Baxter equations and a very short outline of the Thesis. Chapter 1 is devoted to algebraic and differential geometric preliminaries needed for understanding the results of the Thesis. In particular it contains definitions and preliminary, known in the literature, results on Poisson geometry, generalized distributions, Lie algebras and their cohomology, Lie bialgebras and Poisson–Lie groups, Lie systems, and Jacobi structures.

Chapters 2–5 already contain either own results of the Author of the Thesis or obtained in collaboration with co-authors. Chapter 2 based on [dLW20] is devoted to algebraic methods in classification of 3-dimensional Lie bialgebras (the so called coboundary ones, i.e. related with solutions of CYBE or its generalization, modified CYBE). The approach is novel and relies on determining \mathfrak{g} -invariant elements in the Grassmann algebra $\bigwedge^{\bullet} \mathfrak{g}$ by a series of methods. In particular, one of them is exploiting gradings of a Lie algebra and the induced gradings of the Grassmann algebra.

In Chapter 3 (based on [dLW21]) another, more geometric methods are developed for classification of Lie bialgebras. The methods rely on the study of the orbits of the fundamental vector fields of the

action of the automorphism group $\text{Aut}(\mathfrak{g})$ on the space of bivectors $\wedge^2 \mathfrak{g}$. Since the distribution D_x spanned by these vector fields is generalized, i.e. dimension of D_x depends on x , determination of the leaves of the generalized foliation tangent to this distribution is a nontrivial task. A method of the Darboux families is used to solve it, which is based on a novel notion generalizing the notion of the Darboux polynomial. This method is then used to classifying the coboundary Lie bialgebra structures on 4-dimensional indecomposable Lie algebras and on $\mathfrak{gl}(2)$. Additionally, an approach for constructing a faithful matrix representation of a class of finite-dimensional Lie algebras with a non-trivial center is established, which contains the class of indecomposable 4-dimensional Lie algebras with a non-trivial center. This approach is of independent interest and is described in Appendix B.

In Chapter 4 (based on [CdLW22]) a generalization of Lie systems, foliated Lie systems, is introduced and studied. These last are nonautonomous systems of first order ODEs on a manifold M described by a t -dependent vector field $X = \sum_{i=1}^r g_i X_i$, where X_1, \dots, X_r are vector fields on M spanning an r -dimensional Lie algebra and tangent to a foliation \mathcal{F} on M , while g_i are t -dependent functions on M constant along X_j for any fixed t (one retrieves the notion of “usual” Lie system if g_i are constant over M). Foliated Lie systems are studied in detail, in particular a “foliated” version of the Lie-Scheffers theorem is proven stating the existence of generalized (foliated) superposition rule. Several examples are considered among which there is one related to an r -matrix on a Lie algebra \mathfrak{g} : the corresponding foliated Lie system then appears on the corresponding Lie group. Also, a new class of the so-called generalized Ermakov systems is obtained within the formalism.

Chapter 5 contains unpublished yet results related to a generalization of method of deformations of Lie Hamilton systems (i.e. Lie systems generated by vector fields X_i which are hamiltonian with respect to some Poisson structure) elaborated previously in [13]. The generalization mentioned is intended to cover important cases of Lie systems, which cannot be endowed with the Lie-Hamilton system structure (such is the case of the Schwarz equation [49]). Chapter 5 rather describes general ideas which are subsequently illustrated by quite interesting examples. A central role in these examples is played by one parameter family of Poisson bivectors obtained on the dual Lie bialgebra \mathfrak{g}^* to a Lie bialgebra \mathfrak{g} . It is shown that in the example of the Lie system related to the Schwarz equation (and to the Lie algebra $\mathfrak{sl}(2)$) the corresponding vector fields X_i are hamiltonian with respect to a certain Jacobi structure. Finally, a one parameter family of “Lie-Jacobi” systems deforming the initial one related to the Schwarz equation is obtained.

In Chapter 6 “Conclusions and Outlook” one finds a discussion of limits of applicability of methods of the Thesis and possible perspectives.

In Appendix A several commands and procedures designed for the Mathematica symbolic computation software are described, which were used in the results of Chapters 2 and 3.

3 Critical remarks

- One of the weakest points of the Thesis is the absence of a comprehensive overview of the results which would serve as a guide for the reader over the very complicated body of the Thesis. Unfortunately “Outline of the thesis” contained in Introduction is too short and can not serve as such a guide.
- Also, an index of exploited notations would simplify the reader’s efforts. Otherwise, it would be desirable to add references to previously introduced notations in some places, for instance in Theorem 3.1.3 the reference to notations $V_{\mathfrak{g}}^{(2)}$ and $V_{\mathfrak{g}}^{(3)}$.

- Chapter 4 of the Thesis based on article [CdLW22] practically coincides with Sections 3-8 of this article. However, since the main notion of the Thesis are Lie bialgebras, it would be desirable to make the exposition of the examples related to Lie bialgebras more careful and transparent. For instance, in Example 31 at p.110 it is not clear what does the dichotomy “if \mathfrak{g}^{lp} is a matrix Lie algebra” vs “if \mathfrak{g}^{lp} is a not matrix Lie algebra” mean: does it concern the previously introduced Lie algebra \mathfrak{g}^{lp} , given explicitly by commutation relations, or a general Lie algebra.
- On the other hand, the historical overview in Introduction although well written, does not contain any information related to Lie systems, which are essential ingredient of the Thesis.
- Results on classification of solutions of CYBE obtained by Belavin and Drinfeld in [35,36] are traditionally referred to as the Belavin–Drinfeld results and cannot be addressed to Drinfeld alone, as it was done in Introduction.
- Chapter 2 of the Thesis contains purely algebraic results which is in a slight contradiction with the title of the Thesis, which probably would be more adequate if read “Geometric and algebraic approaches to Lie bialgebras, their classification, and applications”.
- In the 3-rd paragraph at p. 26 it is stated that “this thesis is solely devoted to triangular r -matrices”, which is not true, since the quasitriangular ones are studied as well.
- The first sentence in the second paragraph of Sect. 5.1.1 is unclear unless put at the end of this paragraph.
- The notion of a minimal Lie algebra used in the proof of Prop.5.3.2 was not defined previously.
- List of found misprints is presented below.

4 Bibliography of Daniel Wysocki

This Thesis is based on 3 publications of its Author with co-authors in highly renowned in the field of mathematics and mathematical and theoretical physics journals:

- [dLW20] (1 citation excluding self-citations), *J. Lie Theory* - 70 points in the List of scientific journals and peer-reviewed materials from international conferences of the Polish Ministry of Education and Science, 2023 (List for short);
- [dLW21] (1 citation excluding self-citations), *Symmetry* - 70 points in the List;
- [dLW21] *J. Phys. A: Math. Theor.* - 100 points in the List.

Besides, Mr Daniel Wysocki is one of the authors of the preprint “A note on cohomology for multiplier Hopf algebras”, <https://arxiv.org/abs/1908.01033> (yet unpublished), which is not directly related to the topic of the Thesis.

5 Conclusion

The Thesis is generally well written (see however critical comments above). A separate praise should be addressed for carefully prepared figures (and schemes depicting Darboux trees in Chapter 3), which help the reader a lot. It contains an exhaustive list of references, showing the wideness of considered problems and their place in the scientific literature.

The Thesis contains new interesting mathematical results, which already have or may have in future applications in mathematics in physics. In particular, approaches to classification of Lie bialgebras of Chapters 2 and 3 hopefully would serve as a first step to a more important (and complicated) matter of classification of infinite-dimensional Lie bialgebra structures on $\mathfrak{g}[\lambda^{-1}, \lambda]$, or alternatively, to equally important matter of classification of solutions of AYBE (finite- or infinite-dimensional).

The Author of the Thesis displays great mathematical crudition and creativity which allows him to attack complicated algebraic and differential geometric problems of wide spectrum (in particular, symbolic computation software is mastered).

The Thesis is bases on 3 publications of its Author in highly renowned in the field of mathematics and mathematical and theoretical physics journals (correspondingly 70, 70, and 100 points in the List of Polish Ministry of Education and Science), which although being very fresh (years 2020, 2021, 2022) already have citations (not being self-citations). Besides, he is an author of one unpublished preprint not directly related with the topic of the Thesis, which shows the wideness of his scientific interests.

Summarizing, in my opinion the Author of this Doctoral Thesis Mr Daniel Wysocki satisfies all the requirements and deserves to be awarded the doctoral degree in natural sciences in the discipline of mathematics.

6 List of misprints

- p. 13 : in the 3-rd par. two different characters are used for the space of vector fields.
- p. 13. the last formula in 2-nd par. of Sect. 1.1 : $[a, c]_{T(\mathfrak{g})} \rightarrow [a, c]_{T(\mathfrak{g})} \otimes b$
- p. 15, last sent. in 1-st par. od Sect. 1.2 : *differentiables* \rightarrow *differentiable*
- p. 16, sent. before Theorem 1.2.1 : *Fröbenius* \rightarrow *Frobenius*
- p. 24. third displayed formula from below : $==\rightarrow=$
- p. 25, Def. 1.5.4 : $\otimes^2 \text{ad}_v w \rightarrow \otimes^k \text{ad}_v w$
- p. 25, Def. 1.5.5 : $\dots \alpha_{\sigma_k} \rightarrow \dots \otimes \alpha_{\sigma_k}$
- p. 51, a par. after Theor. 2.7.2 : $\lambda^m \mathfrak{g} \rightarrow \Lambda^m \mathfrak{g}$; this mistake is also present at p. 52 in the proof of Prop. 2.8.1 and at p. 139 in the 2-nd par.
- p. 110, 1-st sent. of Ex. 31 : $\mathbb{R} \rightarrow \mathbb{R}^n$
- p. 123, 1-st sent. : *simply and simply connected* \rightarrow *connected and simply connected*
- p. 123, 1-st sent. after 1-st displayed formula : $= 2X_i^R \rightarrow = 2X_\alpha^R$

- p. 127, the last sentence : $R_{g^{-1}*g} \rightarrow R_{g^{-1}*}$
- p. 130, Prop. 5.2.1 : $\Lambda^K \rightarrow \Lambda_K$ - the notation should be unified
- p. 132, 1-st sent. after 1-st displayed formula : *Recalls* \rightarrow *Recall*
- p. 134, 1-st sent. of 2-nd par. : *shows* \rightarrow *show*
- p. 137, 1-st sent. of 2-nd par. : *there several* \rightarrow *there are several*
- p. 138 : there is an empty citation in the one by last par.
- p. 151 : the name of the journal should be abbreviated in reference [83] (as in [82])

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References

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