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To whom it may concern

Report on Bartłomiej Polaczyk's PhD thesis "Concentration of measure and functional inequalities"

Introduction. Bartłomiej Polaczyk is a PhD student at the University of Warsaw under the supervision of Radosław Adamczak. He works at the interface of analysis and high dimensional probability, on problems mainly related to the concentration of measure phenomenon, its analytic aspects as well as various applications. He has an impressive research record comprising 6 publications, 4 of which have already been published in leading and top journals in the field: *Journal of Functional Analysis, Electronic Journal and Electronic Communications in Probability, Transactions on Machine Learning Research.* His 2 single-authored publications strongly attest to his independence and maturity as a researcher, whilst another 4 prepared in collaboration with his supervisor as well as junior faculty also demonstrate collaborative skills. It needs to be commended that his work has already received around 30 citations (according to Google Scholar), astounding at his early stage.

Thesis content. Bartłomiej's thesis is devoted to an in-depth study of certain functional inequalities and their intimate links to concentration of measure, a challenging model of binary random vectors with dependent coordinates, and bounds for suprema of stochastic processes based on sampling (the so-called Hoeffding statistics).

Let $1 < p \leq 2$ and let μ be an invariant measure for a Markov chain with a generator L and the Dirichlet form $E(f,g) = -\int f(Lg)d\mu$. Neglecting some technical details, we say that μ satisfies the p-Beckner inequality with constant α_p, if

$$\alpha_p \left(\int f^p d\mu - \left(\int f d\mu \right)^p \right) \leq \frac{p}{2} E(f^{p-1}, f),$$

for all functions f from an appropriate class. When $0 < p \leq 2$, it is said that the p-log-Sobolev inequality with constant ρ_p holds if

$$\rho_p \left(\int f^p \log(f^p) d\mu - \left(\int f^p d\mu \right) \log \left(\int f^p d\mu \right) \right) \leq \frac{p^2}{p-1} E(f^{p-1}, f).$$

A limiting version of this as $p \to 1$ is called the modified log-Sobolev inequality with constant $\rho_1 > 0$ and amounts to

$$\rho_1 \left(\int f \log f d\mu - \left(\int f d\mu \right) \log \left(\int f d\mu \right) \right) \leq E(\log f, f).$$

It is of interest to investigate the hierarchy between these notions. In particular, it has been noted many times and is well-known that the if p-Becker inequalities hold with some constants α_p uniformly bounded away from 0 as $p \to 1+$, then the modified log-Sobolev inequality holds. The main result of Chapter 2 (and [6]) is that, surprisingly, the reverse implication holds, and even more, the optimal asymptotic relationship between the constants in both inequalities has been derived, which is quite satisfactory, essentially closing the "bigger" picture of this topic. This has also significant implications for concentration of measure and immediate novel applications to various stochastic models, including random permutations, zero-range processes, strong Rayleigh measures, exponential random graphs, and geometric functionals on Poisson path space.

Chapter 3 (based on unpublished results) is closely related. The author answers in negative a problem recently posed by Mossel, Oleszkiewicz and Sen regarding the implications between p-log-Sobolev inequalities for the interval $(0, 1)$. It has been elusive whether there are any nontrivial subintervals $I \subset (0, 1)$ such that for *all* p, q in I, we have $\alpha_p \geq c\alpha_q$ with a positive constant $c = c_{I,\mu}$ depending only on I (and of course μ). Mossel, Oleszkewicz and Sen have shown that on $(1, 2]$ this indeed is true, and $\alpha_p = \alpha_q \frac{qq'}{pp'}$ for all $1 < q \leq p \leq 2$, where p' and q' are the Hölder conjugates to p and q, respectively. Bartłomiej's main result gives a strong counter-example showing a striking difference to this nice behaviour: for every interval $I \subset (0, 1)$, there are $p, q \in I$, $p > q$ and a measure μ for which the q-log-Sobolev inequality holds whilst the p-log-Sobolev inequality fails. The examples come from birth-death processes on the positive integers. En route to his result, Bartłomiej develops highly nontrivial and intricate conditions characterising the p-log-Sobolev inequalities for such processes, which are certainly of interest in their own right.

There is a plethora of very precise concentration-of-measure-type results for random vectors with independent coordinates (i.e., for product measures). For many important (practical) models, imposing a product structure is too restrictive and the theory is *not* that well-developed for random vectors with dependencies. One pioneering work in this direction is of Pemantle and Peres who have introduced the notion of stochastic covering property, SCP. This in turn is a relaxation of the so-called strong Rayleigh property, SRP, developed in the seminar work of Borcea, Brändén and Liggett (in the context of geometry of multivariate complex polynomials), surprisingly widely applicable in probabilistic setting, as a source of concentration. It is known that SRP implies SCP and that the converse is false in general. Pemantle and Peres have shown that every SCP measure μ on the boolean cube $\{0,1\}^n$ enjoys a sub-Gaussian concentration: $\mu(f > \int f d\mu + t) \leq \exp(-t^2/(8n))$ for every function $f \colon \{0,1\}^n \to \mathbb{R}$ which is 1-Lipschitz with respect to the Hamming distance $d_H(x,y) = \sum_{i=1}^n \mathbf{1}_{x_i \neq y_i}$. The main result of Chapter 3 (and [5]) extends this to weighted Hamming-type distances $d_\alpha(x,y) = \sum \alpha_i \mathbf{1}_{x_i \neq y_i}$. It turns out that Pemantle and Peres' argument is robust enough to handle this more general setting. The same approach also yields an analogous concentration bound for matrix-valued functions, extending a recent result of Aoun, Banna and Youssef. Furthermore, in the same matricial setting, the authors derive a generalisation of the Bernstein-type bounds for linear combinations of positive definite matrices due to Kyng and Song. The rest of this chapter establishes interesting new and rather delicate precise estimates for Bernoulli random vectors with independent components conditioned on having a fixed sum – generalising the uniform measure on a *slice* of the boolean cube. This is a principal model of mildly dependent components which has been actively studied in the recent years by many prominent researchers (such as Bobkov, Filmus, O'Donnell Quastel, Tetali). Nevertheless many basic questions remain open, e.g. Khinchin inequalities with sharp constants.

The last chapter of Bartłomiej's dissertation concerns estimates on suprema of stochastic processes arising from statistics of sampling. This topic continues a rich and long line of investigation going back to deep and influential works of Talagrand. Suppose S is a fixed collection of vectors in the symmetric cube $[-1,1]^n$. Let σ be a permutation on $\{1,\ldots,n\}$ chosen uniformly at random. Fix $m \leq n$ and let $Z = \sup_{x\in S}\sum_{k=1}^{m} x_{\sigma_k}$. This is a particular form of the so-called Hoeffding statistics stemming from and vital in certain applications. Bartłomiej's main result is the following Bennett-type concentration inequality,

$$\mathbb{P}(Z \geq \mathbb{E}Z + t) \leq 2\exp\left(-\frac{t}{36}\log\left(1+\frac{t}{46v}\right)\right), \ t > 0$$

with the variance proxy $v = \mathbb{E}\sup_{x\in C}\sum_{k=1}^{m} x_{\sigma'_k}^2$, where $\sigma'_1, \dots, \sigma'_n$ are i.i.d. uniform on $\{1, \dots, n\}$ (corresponding to sampling *with* replacement). This improves in some situations upon earlier results by Chatterjee, by Bercu, Deylon and Rio and by Albert.

Assessment. The main results of Bartłomiej's thesis establish several important probabilistic inequalities rooted in long-standing, competitive and challenging research programmes. Their proofs, in addition to being novel with strong potential of inspiring further development, overcome substantial technical difficulties which are likely to arise in future investigations. This is a well-organised, reader-friendly, well-written thesis. Undoubtedly, it merits PhD degree.

Summary. Bartłomiej is a productive PhD student, with very solid analytic and probabilistic background. He has obtained several important and deep results, clearly presented in his comprehensive, excellent PhD thesis. It strongly demonstrates his creativity, vast knowledge, as well as technical prowess. It is a great pleasure for me to attest that his PhD thesis fulfills and exceeds the standard expectations and requirements. **I strongly and wholeheartedly recommend Bartłmiej for a PhD degree with a *distinction* ("wyróżnienie").**

Yours faithfully

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