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# Report on Anna Nenca's PhD thesis, entitled *"Oriented colouring of 2-dimensional grids"*

Anna Nenca's PhD thesis mainly deals with colourings and homomorphisms of oriented graphs, that is, antisymmetric and loopless digraphs. A homomorphism from a graph G to a graph H is an edge preserving mapping from the set of vertices of G to the set of vertices of H (that is, each edge of G has to map to an edge of H). This notion captures the notion of graph colouring, since a (proper) k-colouring of G is nothing but a homomorphism from G to the complete graph $K_k$ of order k. The notion of graph homomorphism readily extends to oriented graphs, and corresponds to the so-called oriented colourings, as introduced by Courcelle in 1994. The oriented chromatic number of an oriented graph G is then defined as the smallest order of a tournament to which G admits a homomorphism. The last chapter of the thesis concerns homomorphisms of signed graphs. A signed graph $(G,\sigma)$ is an undirected graph with positive and negative edges (given by the mapping $\sigma$), two signed graphs being equivalent if one can be obtained from the other by a sequence of vertex-switches (that is, negating the signs of all edges incident with a given vertex). There exists a homomorphism of $(G,\sigma)$ to $(H,\pi)$ if and only if there exists a signed graph $(G,\sigma')$, equivalent to $(G,\sigma)$, and a sign-preserving homomorphism from $(G,\sigma')$ to $(H,\pi)$. Similarly as before, the signed chromatic number of a signed graph $(G,\sigma)$ is defined as the smallest order of a signed tournament to which $(G,\sigma)$ admits a homomorphism.

This thesis focuses on oriented colourings of four particular classes of oriented graphs, namely square grids, cylindrical grids, toroidal grids and strong grids, and on signed colourings of square grids.

Let $P_n$ and $C_n$ denote the path and the cycle of order n, respectively. The square grid G(m,n) is then defined as the Cartesian product $P_m \square P_n$, the cylindrical grid Cyl(m,n) as the Cartesian product $C_m \square P_n$, the toroidal grid T(m,n) as the Cartesian product $C_m \square C_n$, while the strong grid $G^{\boxtimes}(m,n)$ is defined as the strong product $P_m \boxtimes P_n$. In 2003, Fertin, Raspaud and Rowchodhury proved that the oriented chromatic number of every oriented square grid is at most 11 and conjectured that this value could be decreased to 7 (they settled the conjecture for $m \leq 3$). In 2004, Szepietowski and Targan proved that this conjecture also holds for $m = 4$, and that the value 7 cannot be improved when $m = 3$ and $n \geq 7$, or when $m = 4$ and $n \geq 5$. On one other hand, Marshall proved in 2016 that the oriented chromatic number of every oriented cylindrical grid is at most 11.

Concerning oriented colourings of strong grids, Aravind, Narayanan and Subramanian proved in 2011 that the oriented chromatic number of every oriented strong grid $G^{\boxtimes}(2,n)$ (resp. $G^{\boxtimes}(3,n)$) is at most 11 (resp. at most 67) and that there exists such strong grids with oriented chromatic number at least 8 (resp. at least 10). Sopena proved in 2012 that the oriented chromatic number of every oriented strong grid is at most 126.

The study of homomorphisms of signed grids is a more recent topic. Bensmail proved in 2019 that the signed chromatic number of every signed square grid is at most 11. This bound was then decreased to 9 by Dybizbański in 2020.

The study of graph homomorphism problems is a very active area of Graph Theory. In particular, determining the smallest graph to which every graph of a given class admits a homomorphism is a key issue, and this thesis addresses this problem for several graph classes.

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Anna Nenca's thesis comprises nine chapters. **Chapter 1** is a general introduction to the thesis, presenting a few results from the literature on colourings of oriented graphs and of signed graphs, with a special focus on 2-dimensional grids, and announcing the contents of the following chapters. **Chapter 2** is devoted to terminology and notation, and introduces the graph theoretic notions needed for reading the thesis. A few basic algorithms that will be used later are also introduced. Anna Nenca's contributions are then described in **Chapters 3 to 8**, and **Chapter 9** proposes a few directions for future research. The thesis also contains a bibliography gathering 55 references.

In **Chapter 3**, Anna Nenca disproves the conjecture of Fertin, Raspaud and Rowchodhury, by exhibiting an orientation of the square grid G(7,162) that requires at least 8 colours to be properly coloured. The proof highly relies on computer programs and the "trap oriented grid" is built by combining a base oriented grid — that admits homomorphisms to only nine targets tournaments on 7 vertices —, with nine other oriented grids, each of them failing to admit a homomorphism to one if these nine tournaments.
The results presented in this chapter have been published in an international journal[1].

In **Chapter 4**, it is proved that every oriented square grid G(m,n) with $m \leq 8$ admits a homomorphism to a tournament $H_{10}$ of order 10 — obtained by removing one vertex from the Paley tournament of order 11 — thus decreasing by one the general upper bound given by Fertin, Raspaud and Rowchodhury. Again, the fact that every orientation of a square grid with 8 rows admits a homomorphism to $H_{10}$ is proved by computer. The main idea is to prove that for every orientation of the "i-th column", there exists a homomorphism of the $i - 1$ first columns to $H_{10}$ that can be extended to the next column. The existence of such homomorphisms, that are "extendable" from one column to the next one, is proved by means of a computer program, taking advantage of the structure of the automorphism group of $H_{10}$.

The previous upper bound is decreased to 9 in **Chapter 5** for oriented square grids G(m,n) with $m \leq 5$. More precisely, Anna Nenca proves that every orientation of such a square grid admits a homomorphism to the tournament $H_9$, obtained from the Paley tournament of order 7 by adding two new vertices, a source and a sink. As an intermediate step, she proves that every oriented square grid on five rows, such that no

[1] Janusz Dybizbański and Anna Nenca, Oriented chromatic number of grids is greater than 7, *Information Processing Letters* 112 (2012) 113–117.

vertex in the first, third or fifth row is a source or a sink, admits a homomorphism to the Paley tournament of order 7. The proof is based on a computer program, but proving its correctness is more involved than in the previous chapters. Finally, it is also proved that there exists an orientation of the square grid G(7,28) that does not admit any homomorphism to $H_9$.

Cylindrical and toroidal grids are considered in **Chapter 6**. Anna Nenca first gives a short proof of Marshall's result, which states that the oriented chromatic number of every oriented cylindrical grid is at most 11 (this result was indeed a consequence of a more general result of Marshall). She then proves, using computer, that when $m \leq 7$, the oriented chromatic number of every oriented cylindrical grid Cyl(M,n) is at most 10. She finally proves that the oriented chromatic number of every oriented toroidal grid T(m,n) is at most 27, and decreases this upper bound for each value of m in {3,...,7}.
Some of the results presented in Chapters 5 and 6 are gathered in an article that has been accepted for publication in an international journal[2].

**Chapter 7** deals with the oriented chromatic number of strong grids. Anna Nenca proves that there exists an orientation of the strong grid $G^{\boxtimes}(2,398)$ that requires 11 colours to be properly coloured, which gives that the upper bound given by Aravind, Narayanan and Subramanian is optimal, and which improves the previous known lower bound. Concerning upper bounds, Anna Nenca proves that the oriented chromatic number of every oriented strong grid is at most 88, using a direct combinatorial proof. Furthermore, it is proved that every orientation of every strong grid of the form $G^{\boxtimes}(3,n)$ (resp. $G^{\boxtimes}(4,n)$) admits a homomorphism to the Paley tournament of order 19 (resp. to an oriented graph of order 38 obtained by connecting in a specific way two copies of the Paley tournament of order 19). The proofs of both these results rely on a computer program, using the ideas that have been developed in the previous chapters for square grids.
Some of the results presented in Chapters 5 and 7 have been published in an international journal[3]. However, some upper bounds presented in this paper are improved in this chapter.

Finally, the chromatic number of signed square grids is studied in **Chapter 8**. Since the acyclic chromatic number of square grids is at most 3, it follows from a result of Ochem, Pinlou and Sen (2017), that this parameter is bounded from above by 6. Anna Nenca first gives a direct proof of this bound. She then proves that the chromatic number of every signed grid with $3 \leq m \leq 7$ rows and $n \geq 4$ columns is at most 5 (by means of a computer aided proof), that the chromatic number of every signed grid with two rows is at most 4, and that both these bounds are tight.
The results presented in this chapter have been published in an international journal[4].

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**Conclusion.** The manuscript is clearly written, in a concise way, and contains an interesting bunch of results concerning homomorphisms of various types of grid graphs. Several results are obtained by means of computer checking, some others rely on purely

[2] Anna Nenca, Oriented chromatic number of Cartesian products $P_m \square P_n$ and $C_m \square P_n$, *Discussiones Mathematicae Graph Theory* (2020), in press.
[3] Janusz Dybizbański and Anna Nenca, Oriented chromatic number of Cartesian products and strong products of paths, *Discussiones Mathematicae Graph Theory* 39 (2019), 211-223.
[4] Janusz Dybizbański, Anna Nenca, Andrzej Szepietowski, Signed coloring of 2-dimensional grids, *Information Processing Letters* 156 105918, (2020).

combinatorial proofs. Almost all the problems considered in this thesis were previously studied by various authors, and Anna Nenca often improved those existing results.

Most of the results gathered in this thesis have been published, leading to four papers in international journals (*Information Processing Letters* 2012 & 2020, *Discussiones Mathematicae Graph Theory* 2019 & 2020), Anna Nenca being the sole author of one of them.

In view of all of this, I consider that the submitted thesis meets the usual requirements for a PhD thesis, and I recommend that it can be defended with the aim of awarding a PhD degree.

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