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Warsaw, August 15, 2020.

Reviewer's report on the revised PhD dissertation
***Oriented coloring of 2-dimensional grids* by Anna Nenca**

The dissertation focuses on oriented coloring of graphs. This is a natural analogue of classical graph coloring for directed graphs (digraphs). Classical coloring of a given graph G using k colors can be understood as a homomorphism from G to the clique K_k. Equivalently, we could ask for a homomorphism to *any* k-vertex graph. Similarly, a digraph G admits *oriented coloring* in k colors if there is a homomorphism from G to any k-vertex digraph. Again, equivalently we can focus on homomorphisms to digraphs with an edge between any pair of vertices, called tournaments. Unlike in the undirected case, there are $2^{\binom{k}{2}}$ tournaments on k vertices, so the problem is much more complex than the classical graph coloring. The results of the thesis are stated in terms of *undirected* graphs. The oriented chromatic number $\overrightarrow{\chi}(G)$ of an undirected graph G is the maximum oriented chromatic number of its orientations, i.e., digraphs obtained from G by replacing every edge uv either by the arc (u, v) or the arc (v, u).

Oriented chromatic number has been introduced in 1994 by Courcelle and since then it has been studied in at least 70 research articles, some of them authored by top graph theorists. Most of these works focus on bounding $\overrightarrow{\chi}(G)$ for various classes of undirected graphs G. In 1994 Raspaud and Sopena showed that $\overrightarrow{\chi}(G) \leq 80$ for every planar graph G. Despite many efforts, this bound has not been improved yet, hence many authors study *subclasses* of planar graphs, like outerplanar graphs or planar graphs with a lower bound on the girth. In 2003 Fertin, Raspaud and Royhowdhurry considered *grids*, one of the simplest planar graphs, showing that $\overrightarrow{\chi}(G) \leq$

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11 for every grid G. However, they were unable to find a grid which needs 8 colors, and hence they conjectured that 7 colors suffice.

Results

Chapter 3 presents an example of an oriented grid which cannot be colored by 7 colors, refuting the conjecture of Fertin et al. The result is obtained by means of a computer program. The program examined all 456 non-isomorphic tournaments on 7 vertices, and for each H among them found a *trap*, i.e., an oriented grid G_H which does not admit a homomorphism to H. The final counterexample is a merge of all traps. A trap for a tournament H is found by a surprising heuristic algorithm: begin with an oriented path on 7 vertices (a 7×1 grid) and generate all its colorings. Then, extend the path to a 7×2 grid and choose the orientation of the new edges so that the number of colorings is minimized (by checking all orientations and all colorings that extend the colorings of the previous grid). This step is repeated to generate oriented grids of size $7 \times 2, 7 \times 3, \ldots$ until the number of colorings drops to 0. This result was published in a joint work with J. Dybizbański in Inf. Proc. Lett.

In chapters 4 and 5 the author studies grids with one dimension fixed, namely grids $m \times n$ with $m \in \{5, 6, 7, 8\}$. Chapter 4 proves that 10 colors suffice for $m = 8$ (which implies the same for $m < 8$). Fertin et al. proved that 7 colors suffice when $m = 3$ by showing that every coloring of an orientation G of $m \times n$ can be extended to any $m \times (n + 1)$ oriented grid obtained by extending G. To this end, they show (by hand) that every coloring of the last column of G can be extended to the new edges. In the thesis a similar method is applied, but instead of considering all colorings of the last column, a set of colorings S^*_{max} is found such that every coloring of the last row in the smaller grid that belongs to S^*_{max} can be extended to a coloring which restricted to the (new) last column again is in S^*_{max}. The set is obtained by a computer program which begins with the set of all colorings and removes the colorings which cannot be extended, until all of them can. Although some symmetries in the coloring were exploited so that the computation time was reduced up to 25 times, the author reports (on her website) that it took more than a week (on a single processor) to compute S^*_{max}. The result from Chapter 4 is contained in an author's article accepted

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for publication in Discuss. Math. Graph Th. (DMGT).

Chapter 5 proves that only 9 colors suffice for $m = 5$, i.e., grids with five rows. Here, the proof follows (and simplifies a bit) the approach of Szepietowski and Targan (Inf. Proc. Lett., 2004) who showed that 7 colors suffice for $m = 4$. As is common in the litetature, Szepietowski and Targan map their $4 \times n$ grids to so-called Paley tournament of order 7, denoted $\overrightarrow{T}_7$. Paley tournaments seem to be good candidates for homomorphic images because they are highly symmetric. The approach in Chapter 5 is to use $\overrightarrow{T}_7$ again. It turns out that this is always possible, unless the colored oriented grid has vertices of indegree or outdegree 0 in the second or fourth row. To color these special vertices, two new vertices are added to $\overrightarrow{T}_7$ (one with indegree 0 and one with outdegree 0). This result was published in a joint work with J. Dybizbański in DMGT.

In Chapter 6 the author studies cylindrical grids (products of a cycle and a path) and toroids (products of two cycles, aka torus grid graphs). An interesting result here is the upper bound of 27 colors for oriented coloring of toroids. This class of graphs has not been studied before in the context of oriented chromatic number. The result is obtained by providing a homomorphism to so-called Tromp graph, a highly symmetrical digraph used by many previous papers on oriented coloring. More precisely, all but two layers of vertices form a cylindrical grid and they are colored with 11 colors using a known result; next the author carefully selects colors for the two remaining layers using known properties of the Tromp graph. The chapter contains also a new proof of the previously known upper bound for cylindrical grids. The author claims that it is 'simpler and more direct', though in the opinion of the reviewer the previous argument by Marshall was also rather simple and direct (and, contrary to the present proof, did not use a computer). The chapter has also a few minor results: a lower bound for cylindrical grids (an immediate application of a lower bound for a grid) and improved bounds for cylindrical grids and toroids with a small number of layers, obtained by using the computer-based approach of Chapter 4. Some results from Chapter 6 are contained in an author's article accepted for publication in DMGT.

Chapter 7 presents bounds on oriented chromatic number for strong grids,

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i.e., grids with two diagonals put in every internal face. This looks like a rather exotic class of graphs, though one can interpret them as a strong product of two paths. Most of the results in Chapter 7 are obtained by reusing the methods from chapters 3, 4 and 5. A somewhat more interesting part is Theorem 7.1, which shows that 88 colors suffice for strong grids. This improves a previous result by Sopena from 2012 (126 colors). The goal is achieved by using the Tromp graph and coloring the grid layer by layer. Some results from Chapter 7 are published in a joint work with J. Dybizbański in DMGT.

In Chapter 8 the author studies a different, but related problem of signed coloring, i.e., homomorhpisms of 2-edge-colored graphs. Not surprisingly, the tools of Chapter 5 can be again applied here to get some bounds for grids with few rows. An interesting result in this chapter is an elegant proof of a lower bound of 5 colors for the class of grids. The results from Chapter 8 are published in a joint work with J. Dybizbański and A. Szepietowski in Inf. Proc. Lett.

Evaluation of the Results

Given the lack of progress in improving the bound on oriented chromatic number for planar graphs, studying grids, which are simple and highly structured planar graphs, is well motivated. The author obtains claims on infinite families of objects using computer search, which is interesting and a bit surprising, though not entirely novel because similar methods were applied to these problems by Szepietowski and Targan [IPL 2004].

In the opinion of the reviewer, the most interesting result of the thesis is the new lower bound on the oriented chromatic number of the class of grids (Chapter 3). The result builds on the previous attempt of Szepietowski and Targan but introduces a new local search method. This result is an important contribution to the theory of oriented coloring and it will be cited in future papers focusing on planar graphs. Other valuable contributions are the new upper bounds of the oriented chromatic number for toroids and strong grids, as well as the lower bound for signed chromatic number of grids.

Determining the oriented chromatic number of "thin grids", i.e., grid graphs

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with one dimension fixed to 5 or 8 (or cylinders/toroids with few layers) seems less motivated. To the taste of the reviewer these classes of graphs are too narrow, and the interest in the results is very limited. The approach of finding the set S^*_{max} used in Chapter 4 is new but rather natural, while the approach in Chapter 5 is an adaptation of earlier work. The results on thin grids in the remaining chapters just reuse these methods. Overall, although the results for thin grids constitute a contribution to the oriented coloring problem, they seem to be below the bar for PhD theses.

Writing style

Most of the time the writing style is clear and apart from a few wrong articles the language quality is fine. An exception is perhaps Section 6.1, where the statements of Lemmas 6.8 and 6.9 lack mathematical precision, and one can understand their meaning only by matching their application in the proof of Theorem 6.1.

In a few places I would appreciate more comments on some decisions taken, for example why graphs similar to Paley tournaments are used, or how important (for the running time of the computer programs) was exploiting the symmetries in the homomorphic images. Also, a more explicit credit should be given to the work of Szepietowski and Targan [IPL 2004], especially in Chapter 5. Similarly, it would be nice to point out the origin of the technique of oriented coloring with Tromp graphs using so-called properties $P_c(i, j)$ in chapters 6 and 7.

The difference between the first and the revised version

One source of criticism in my review of the first version was that the nature of the proofs there, based on computer processing, reveals very little structure of the problem studied. I am happy to see that the author took an effort to learn some proof techniques used in the literature, in particular a method of coloring with Tromp graph which resulted in some new upper bounds proven by hand, including an improvement of a bound by Sopena.

The other critical remark was that most of the first version of the thesis was devoted to thin grids, which seem to be embarrassingly narrow graph classes. In the revised version there is more balance between the results for thin grids and somewhat more general graphs, thanks to adding the upper bounds for toroids and strong grids and the lower bound for the signed coloring of grids.

Conclusion

A PhD thesis should present an original solution to a research problem. This criterion is mostly fulfilled by the contents of Chapter 3 which proposes an original approach to an open problem studied before and applies it with success. For the remaining results no original proof techniques are developed. However, the arguments required some level of technical skill and show some variety of tools used. As a result, the choice of the right tool allowed the author to improve some previous results.

To sum up, in my opinion the revised dissertation **fulfills** the requirements for PhD theses in mathematical sciences formulated by law and accepted in the community.

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