50-th Mathematical Olympiad in Poland

Final Round, April 14–15, 1999

First Day

1. Point D lies on the side BC of the triangle ABC and is chosen so that $AD > BC$. Point E lies on the side AC and satisfies

$$\frac{AE}{EC} = \frac{BD}{AD - BC}.$$

Prove that $AD > BE$.

2. Given positive integers $a_1 < a_2 < a_3 < \ldots < a_{101}$ less than 5050. Prove that there exist four different numbers a_k, a_l, a_m, a_n such that the number $a_k + a_l - a_m - a_n$ is divisible by 5050.

3. Prove that there exist positive integers $n_1 < n_2 < \ldots < n_{50}$ such that

$$n_1 + S(n_1) = n_2 + S(n_2) = n_3 + S(n_3) = \ldots = n_{50} + S(n_{50}),$$

where $S(n)$ denotes the sum of the digits of n.

Second Day

4. Find out for which integers $n \geq 2$ the system of equations

$$\begin{cases} x_1^2 + x_2^2 + 50 = 16x_1 + 12x_2 \\ x_2^2 + x_3^2 + 50 = 16x_2 + 12x_3 \\ x_3^2 + x_4^2 + 50 = 16x_3 + 12x_4 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_{n-1}^2 + x_n^2 + 50 = 16x_{n-1} + 12x_n \\ x_n^2 + x_1^2 + 50 = 16x_n + 12x_1 \end{cases}$$

has integer solutions $x_1, x_2, x_3, \ldots, x_n$.

5. Let $a_1, a_2, ..., a_n, b_1, b_2, ..., b_n$ be given integers. Prove that

$$\sum_{1 \leq i < j \leq n} (|a_i - a_j| + |b_i - b_j|) \leq \sum_{1 \leq i,j \leq n} |a_i - b_j|.$$

6. In a convex hexagon $ABCDEF$ the following equalities hold:

$$\angle A + \angle C + \angle E = 360^\circ, \qquad \frac{AB}{BC} \cdot \frac{CD}{DE} \cdot \frac{EF}{FA} = 1.$$

Prove that $\dfrac{AB}{BF} \cdot \dfrac{FD}{DE} \cdot \dfrac{EC}{CA} = 1$.