46-th Mathematical Olympiad in Poland

Final Round, Gdynia, March 31 – April 1, 1995

First Day

1. Determine the number of the subsets of $\{1, 2, \ldots, 2n\}$, in which the equation $x + y = 2n + 1$ has no solutions.

2. The diagonals of a convex pentagon cut it into a pentagon and ten triangles. What is the largest number of the obtained triangles which may have an equal area?

3. Given is a prime $p > 3$; set $q = p^3$. Define the sequence (a_n) by:

$$a_n = \begin{cases} n & \text{for} \quad n = 0, 1, 2, \ldots, p-1, \\ a_{n-1} + a_{n-p} & \text{for} \quad n > p-1\,. \end{cases}$$

Determine the remainder when a_q is divided by p.

Second Day

4. Numbers $x_1, x_2, \ldots, x_n$ are positive with the harmonic mean equal to 1. Determine the smallest value of

$$x_1 + \frac{x_2^2}{2} + \frac{x_3^3}{3} + \ldots + \frac{x_n^n}{n}\,.$$

5. In the urn there are n sheets of paper labelled $1, 2, \ldots, n$. We draw the sheets one by one without putting them into the urn again. When we obtain a sheet with a number divisible by k, we stop the drawing. For a fixed n, determine all values of k for which the expected value of the number of the drawn sheets is equal to k.

6. Given three rays k, l, m in the space with a common beginning P and a point A, distinct from P, belonging to k. Prove that there exists exactly one pair of the points B and C belonging to l and m respectively, such that

$$PA + AB = PC + CB \qquad \text{and} \qquad PB + BC = PA + AC\,.$$