45-th Mathematical Olympiad in Poland

Final Round, Warszawa, April 10–11, 1994

First Day

1. Determine all triples (x, y, z) of positive rationals, such that the numbers $x+y+z$, $\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}$, xyz are all natural.

2. Given two parallel lines k and l and a circle not intersecting the line k. From the point A lying on k draw two tangents to the given circle that intersect the line l in two points B and C. Let m be the line passing through A and the midpoint of BC. Prove that all the lines m obtained in that way (corresponding to the various points A on k) have a common point.

3. Given an integer $c \geq 1$. To each subset A of $\{1, 2, \ldots, n\}$ there is assigned an integer $w(A)$ from $\{1, 2, \ldots, c\}$, such that the following condition is satisfied

$$w(A \cap B) = \min(w(A), w(B)) \qquad \text{for} \quad A,\, B \subseteq \{1, 2, \ldots, n\}\,.$$

Let $a(n)$ be the number of such assignments. Compute $\lim\limits_{n\to\infty} \sqrt[n]{a(n)}\,$.

Second Day

4. We have three vessels without the scale: the first one (empty) may contain at most m litres, the second one (empty) may contain at most n litres, and the third one (full of water) may contain at most $m+n$ litres. m and n are relatively prime positive integers. Prove that for any $k \in \{1, 2, \ldots, m+n-1\}$, pouring out of one vessel into another, one can obtain in the third vessel exactly k litres of water.

5. $A_1, A_2, \ldots, A_8$ are the vertices of the parallelepiped with the centre O. Prove that

$$4 \cdot \sum_{i=1}^{8} |OA_i|^2 \leq \Big(\sum_{i=1}^{8} |OA_i|\Big)^2 .$$

6. Different real numbers $x_1, x_2, \ldots, x_n$ $(n \geq 4)$ satisfy the conditions:

$$\sum_{i=1}^{n} x_i = 0\,, \qquad \sum_{i=1}^{n} x_i^2 = 1\,.$$

Prove that among these numbers there exist four, say a, b, c, d, such that the inequalities

$$a + b + c + nabc \leq \sum_{i=1}^{n} x_i^3 \leq a + b + d + nabd$$

are valid.