49-th Mathematical Olympiad in Poland

Second Round, February 27–28, 1998

First Day

1. Let $A_n = 1, 2, \ldots, n$. Prove or disprove the following statement: for all integers $n \geq 2$ there exist functions $f : A_n \to A_n$ and $g : A_n \to A_n$ which satisfy
$f(f(k)) = g(g(k)) = k$ for $k = 1, 2, \ldots, n$;
$g(f(k)) = k + 1$ for $k = 1, 2, \ldots, n - 1$.

2. In triangle ABC the angle $\angle BCA$ is obtuse and $\angle BAC = 2\angle ABC$. The line through B and perpendicular to BC intersects line AC in D. Let M be the midpoint of AB. Prove that $\angle AMC = \angle BMD$.

3. a) Assume that nonnegative numbers a, b, c, d, e, f with sum equal to 1 satisfy

$$ace + bdf \geq \frac{1}{108}.$$

Show that

$$abc + bcd + cde + def + efa + fab \leq \frac{1}{36}.$$

b) Do there exist six different positive numbers a, b, c, d, e, f with sum equal to 1 for which the two above inequalities become equalities?

Second Day

4. Find all pairs (x, y) of integers which satisfy the equation $x^2 + 3y^2 = 1998x$.

5. Suppose that nonnegative numbers $a_1, a_2, \ldots, a_7, b_1, b_2, \ldots, b_7$ satisfy

$$a_i + b_i \leq 2 \quad \text{for } i = 1, 2, \ldots, 7.$$

Prove that there exist two different indices $k, m \in \{1, 2, \ldots, 7\}$ for which

$$|a_k - a_m| + |b_k - b_m| \leq 1.$$

6. Prove that in tetrahedron $ABCD$ the edge AB is perpendicular to the edge CD if and only if there exists a parallelogram $CDPQ$ such that $PA = PB = PD$ and $QA = QB = QC$.