47-th Mathematical Olympiad in Poland
Second Round, February 23–24, 1996

First Day

1. Can every polynomial with integer coefficients be expressed as a sum of cubes of polynomials with integer coefficients ?

2. A circle with center O inscribed in a convex quadrilateral $ABCD$ is tangent to the lines AB, BC, CD, DA at points K, L, M, N respectively. Assume that the lines KL and MN are not parallel and intersect at the point S. Prove that $BD \perp OS$.

3. Prove that if $a, b, c \geq -3/4$ and $a + b + c = 1$, then

$$\frac{a}{a^2+1} + \frac{b}{b^2+1} + \frac{c}{c^2+1} \leq \frac{9}{10}\,.$$

Second Day

4. Given a sequence $a_1, a_2, \ldots, a_{99}$ of one-digit numbers with the poperty that if for some n we have $a_n = 1$, then $a_{n+1} \neq 2$; and if for some n we have $a_n = 3$, then $a_{n+1} \neq 4$. Prove that there exist two numbers $k, l \in \{1, 2, \ldots, 98\}$ such that $a_k = a_l$ and $a_{k+1} = a_{l+1}$.

5. Find all integer solutions to the equation

$$x^2(y-1) + y^2(x-1) = 1\,.$$

6. Prove that each interior point of a parallelepiped with edges a, b, c is at most

$$\frac{1}{2}\sqrt{a^2+b^2+c^2}$$

distance off some vertex of the parallelepiped.