46-th Mathematical Olympiad in Poland

Second Round, February 17–18, 1995

First Day

1. Given is a polynomial P with integer coefficients; 2 divides $P(5)$, 5 divides $P(2)$; show that 10 divides $P(7)$.

2. Given is a convex hexagon $ABCDEF$ with $AB = BC$, $CD = DE$, $EF = FA$. Prove that the lines containing the altitudes of the triangles BCD, DEF and FAB taken from the vertices C, E and A respectively, are concurrent.

3. Given are positive irrational numbers a, b, c, d with $a + b = 1$. Prove that $c + d = 1$ if and only if for all positive integers n

$$[na] + [nb] = [nc] + [nd]\,.$$

Note: $[x]$ is the greatest integer not greater than x.

Second Day

4. Positive real numbers $x_1, x_2, \ldots, x_n$ satisfy the condition

$$\sum_{i=1}^{n} x_i \leq \sum_{i=1}^{n} x_i^2\,.$$

Prove that for all real numbers $t > 1$ the inequality

$$\sum_{i=1}^{n} x_i^t \leq \sum_{i=1}^{n} x_i^{t+1}$$

holds.

5. The incircles of the faces ABC and ABD of a tetrahedron $ABCD$ are tangent to the edge AB in the same point. Prove that the points of tangency of these incircles to the edges AC, BC, AD, BD are concyclic.

6. Determine all positive integers n for which the square $n \times n$ can be cut into squares 2×2 and 3×3 (with the sides parallel to the sides of the big $n \times n$ square).