47-th Mathematical Olympiad in Poland

First Round, September–December, 1995

1. Determine all positive integers n, such that the equation $2\sin nx = \tan x + \cot x$ has solutions in real numbers x.

2. A number is called a palindromic number if its decimal representation read from the left to the right is the same as read from the right to the left. Let (x_n) be the increasing sequence of all palindromic numbers. Determine all primes, which are the divisors of at least one of the differences $x_{n+1} - x_n$.

3. In a group of kn persons, each person knows more than $(k-1)n$ others (k, n are positive integers). Prove that one can choose $k+1$ persons from this group so that each chosen person knows all the others chosen.

Note: If a person A knows B, then B knows A.

4. A line tangent to the incircle of the equilateral triangle ABC intersects the sides AB and AC at points D and E respectively. Prove that

$$\frac{AD}{DB} + \frac{AE}{EC} = 1\,.$$

5. Given triangle ABC in the plane such that $\angle CAB = \alpha > \pi/2$. Let PQ be a segment whose midpoint is the point A. Prove that

$$(BP + CQ)\tan\frac{\alpha}{2} \geq BC\,.$$

6. Given two sequences of positive integers: the arithmetic sequence with difference $r > 0$ and the geometric sequence with ratio $q > 1$; r and q are coprime. Prove that if these sequences have one term in common, then they have them infinitely many.

7. Nonnegative numbers a, b, c, p, q, r satisfy the conditions:

$$a + b + c = p + q + r = 1; \qquad p, q, r \leq \frac{1}{2}\,.$$

Prove that $8abc \leq pa + qb + rc$ and determine when equality holds.

8. The ray of light starts from the center of a square and reflects from its sides with the principle that *the angle of reflection is equal to the angle of incidence*. After some time the ray returns to the center of the square. The ray never reached the vertex and has never returned to the center of the square before. Prove that the ray reflected from the sides of the square an odd number of times.

9. A polynomial with integer coefficients when divided by $x^2 - 12x + 11$ gives the reminder $990x - 889$. Prove that the polynomial has no integer roots.

10. Prove that the equation $x^x = y^3 + z^3$ has infinitely many solutions in positive integers x, y, z.

11. In a skiing jump competition 65 contestants take part. They jump with the previously established order. Each of them jumps once. We assume that the obtained results are different and the orders of the contestants after the competition are equaly likely. In each moment of the competition by a leader we call a person who is scored the best at this moment. Denote by p the probability that during the whole competition there was exactly one change of the leader. Prove that $p > 1/16$.

12. Find out whether there exist two congruent cubes with a common center such that each face of one cube has a common point with each face of the other.