Parameterized WQOs, downward closures, and separability problems

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Separability Problems
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The (scattered) subword relation:

$$abba \preceq abracadabra$$
$$verification \preceq oversimplification$$

Notation

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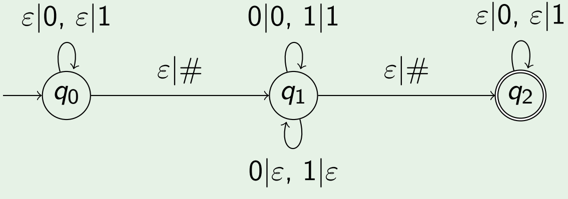

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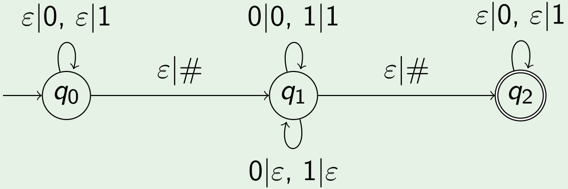

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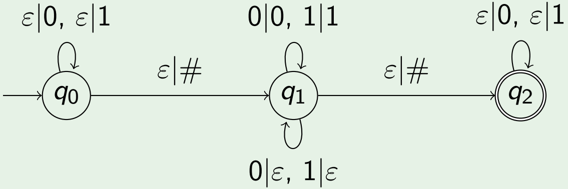

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Definition

- *Rational transduction*: set of pairs given by a finite state transducer.
- For rational transduction $R \subseteq \Sigma^* \times \Gamma^*$ and language $L \subseteq \Sigma^*$, let

$$LR = \{y \in \Gamma^* \mid \exists x \in L : (x, y) \in R\}$$

- A language class $\mathcal{C}$ is a *full trio* if $L \in \mathcal{C}$ implies $LR \in \mathcal{C}$ for such R.

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The *simultaneous unboundedness problem (SUP) for $\mathcal{C}$* is the following:

Given A language $L \subseteq a_1^* \cdots a_n^*$ from $\mathcal{C}$.

Question Does $L{\downarrow} = a_1^* \cdots a_n^*$?

In other words: $\forall k \geqslant 0\colon\ a_1^{\geqslant k} \cdots a_n^{\geqslant k} \cap L \neq \emptyset$?

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- Higher-order pushdown automata (Hague, Kochems, Ong 2016)
- Higher-order recursion schemes (Clemente, Parys, Salvati, Walukiewicz 2016)

Theorem (Goubault-Larrecq, Schmitz 2016)

In any wqo $(X, \leqslant)$ with effective ideals:

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Consequence

If $(\Sigma^*, \leqslant)$ is a wqo with

- effective ideals and
- adherence membership reduces to the SUP,

then for most language classes:

- $\leqslant$-PTL-separability is decidable.

New wqos on words

Simple observation

If $(Y, \leqslant)$ is a wqo and $f\colon X \to Y$, then

$$x \leqslant_f y \iff f(x) \leqslant f(y)$$

defines a wqo on X.

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A transducer is *total unambiguous* if every input word induces exactly one accepting run. It thus defines a function $T\colon \Sigma^* \to \Gamma^*$. Let

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Conjunction

Given wqos $\leqslant_1, \ldots, \leqslant_n$ on X, their *conjunction* is:

$$x \leqslant y \quad \iff \quad \forall i\colon x \leqslant_i y.$$

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Theorem

For each full trio $\mathcal{C}$, the following are equivalent:

- *S-PTL separability is decidable for $\mathcal{C}$ for every order collection S.*
- *The SUP is decidable.*

Example orders I

UFA-defined wqos

Suppose $\mathcal{A}$ is a unambiguous and accepts Σ^*. Define:

$$u \preceq_{\mathcal{A}} v \iff u \text{ is obtained from } v \text{ by "cutting loops"}$$

In other words, $v = u_0 v_1 u_1 \cdots v_n u_n$, such that the run of $\mathcal{A}$ on v loops on each v_i and $u = u_0 \cdots u_n$.

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- $\preceq_{\mathcal{A}}$ is the conjunction of $\preceq_T$ and $\preceq_{T_I}$, and $\preceq_{T_F}$.

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Regular queries

Let $R_1, \ldots, R_k \subseteq \Sigma^*$ be regular. To $w = a_1 \cdots a_n$, we associate the structure

$$\mathcal{M}_w = (\{1, \ldots, n\}, <, P_i, R_j, R_j^{\mathrm{pref}}, R_j^{\mathrm{suf}})$$

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Example: $\mathcal{B}\Sigma_1[<, \mathrm{mod}_d]$ for fixed $d \in \mathbb{N}$.

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Counting-defined wqos

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- Hence, separability by LTT_k decidable if SUP decidable.

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- Let $u \sqsubseteq_{\mathrm{occ},w} v$ if $\mathrm{occ}_w(u) \leqslant \mathrm{occ}_w(v)$. Analogous for $\sqsubseteq_{\mathrm{pref},w}$, $\sqsubseteq_{\mathrm{suf},w}$.
- Let S consist of $\sqsubseteq_{\mathrm{occ},w}$, $\sqsubseteq_{\mathrm{pref},w}$, $\sqsubseteq_{\mathrm{suf},w}$ for all $w \in \Sigma^{\leqslant k}$.
- Then, S-PTL are also known as the k-locally-threshold-testable languages, LTT_k.

- For $\sqsubseteq_{\mathrm{occ},w}$, use transducer $T\colon \Sigma^* \to a^*$ that counts w-occurrences.
- Hence, separability by LTT_k decidable if SUP decidable.
- For CFL, shown by Place, van Rooijen, Zeitoun in 2013 using Presburger arithmetic.

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Ideal representations

Consider a wqo $(Y, \leqslant)$ and $f \colon X \to Y$ and the wqo $(X, \leqslant_f)$. A subset $J \subseteq X$ is an ideal of $(X, \leqslant_f)$ if and only if $J = f^{-1}(I)$ for some ideal I of $(Y, \leqslant)$ such that $f(f^{-1}(I)){\downarrow} = I$.

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- We can therefore use ideals of $(\Gamma^*, \preceq)$ to represent ideals of $(\Sigma^*, \preceq_T)$!
- Ideals of $(\Gamma^*, \preceq)$ are of the shape $X_0^*\{x_1, \varepsilon\}X_1^* \cdots \{x_n, \varepsilon\}X_n^*$.

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Adherence membership

If $I \subseteq Y$ is an ideal with $f(f^{-1}(I))\downarrow = I$, then

$$f^{-1}(I) \in \mathrm{Adh}(L) \quad \text{if and only if} \quad I \in \mathrm{Adh}(f(L)).$$

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Extended adherence membership

If $(\leqslant_s)_{s \in S}$ is a finite family of wqos, then $\mathrm{Adh}_S(L)$ is the set of those families $(I_s)_{s \in S}$ of ideals such that there is a directed $D \subseteq L$ with $I_s = D\downarrow_{\leqslant_s}$.

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If adherence membership for each $\leqslant_s$ reduces to SUP, then this is true for the extended adherence membership problem (product construction).

Proof sketch, part III

Ideal representations for conjunctions

Let $(\leqslant_s)_{s\in S}$ be a finite family of wqos. Let $\leqslant$ be the conjunction of the $\leqslant_s$. Then $I \subseteq X$ is an $\leqslant$-ideal if and only if there is a family of ideals $(I_s)_{s\in S}$ such that $I = \bigcap_{s\in S} I_s$ and $(I_s)_{s\in S}$ belongs to $\mathrm{Adh}_S(I)$.

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Hence, we can again reduce adherence membership to the SUP.

Proof sketch, part IV

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Observation

Let $(\leqslant_s)_{s \in S}$ be a finite family of wqos and let $\leqslant$ be the conjunction of the $\leqslant_s$. Then a language is an S-PTL if and only if it is a $\leqslant$-PTL.

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Theorem (Chaubard, Pin, Straubing, LICS 2006)

For regular languages, definability in $\mathcal{B}\Sigma_1[<, \text{mod}]$ *is decidable.*

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Theorem

For order-2 pushdown languages, separability by $\mathcal{B}\Sigma_1[<, \mathrm{mod}]$ is undecidable.