Separation and Concatenations (2)

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Joint work with Thomas Place

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Summary of Part 1

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2. $\mathcal{C}$ finite $\Rightarrow$ $BPol(\mathcal{C})$-separation decidable.
3. $\mathcal{C}$ finite $\Rightarrow$ $Pol(BPol(\mathcal{C}))$-separation decidable.

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4. For any $\mathcal{C}$,
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This talk: We focus on $BPol(\mathcal{C})$-**separation**.

Separation for $BPol(\mathcal{C})$ when $\mathcal{C}$ is finite

$BPol(\mathcal{C})$-separation: Three main steps

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- We explain their purpose at a high level.

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$\Rightarrow$ we like $Pol(\mathcal{C})$ and hate $BPol(\mathcal{C})$.

Consequence

Even if our goal is $BPol(\mathcal{C})$-separation, we prefer working with $Pol(\mathcal{C})$.

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Meta argument for investigating separation

Learn more on $\mathcal{C}$ to investigate classes built on top of $\mathcal{C}$.

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Learn more on $\mathcal{C}$ to investigate classes built on top of $\mathcal{C}$.

"$\mathcal{C}$-separation decidable $\Rightarrow$ $Pol(\mathcal{C})$-membership decidable."

Recycle this idea to get rid Boolean closure.

New Goal

Reduce $BPol(\mathcal{C})$-separation to a problem for $Pol(\mathcal{C})$.

$BPol(\mathcal{C})$-separation: Three main steps (1)

Step 1

Step 2

Step 3

Getting rid of Boolean closure

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Remarks

- When $n = 2$, we recover the classical notion.
- The longer, the easier to separate.

Boolean closure theorem

$$(L_1, L_2)^k \quad \stackrel{\text{def}}{=} \quad \underbrace{(L_1, L_2, \ldots, L_1, L_2)}_{\text{length } 2k}$$

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Given a lattice $\mathcal{D}$ and two languages L_1, L_2, **TFAE**:

1. L_1 is $Bool(\mathcal{D})$-separable from L_2.
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Mission accomplished!

$\mathcal{D} = Pol(\mathcal{C})$: reduction from $BPol(\mathcal{C})$-separation to a problem for $Pol(\mathcal{C})$.

Boolean closure theorem: proof of $2 \Rightarrow 1$

Induction on k: $(L_1, L_2)^k$ $\mathcal{D}$-separable $\Rightarrow$ L_1 $Bool(\mathcal{D})$-separable from L_2.

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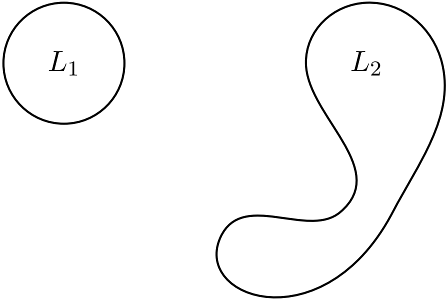

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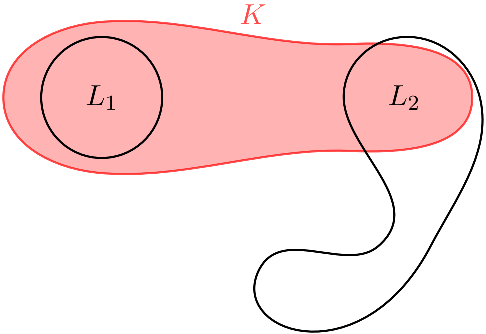

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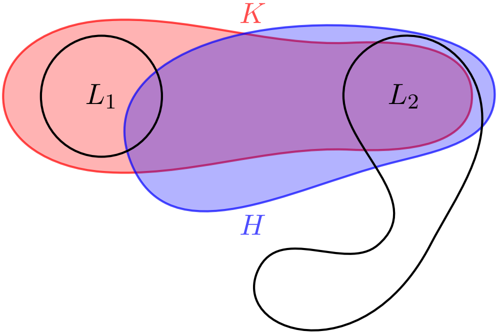

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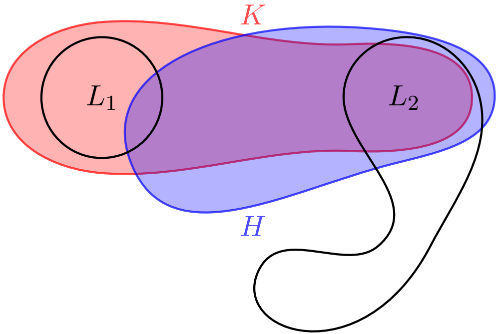

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Induction $\Rightarrow$ $G \in Bool(\mathcal{D})$ separates $L_1 \cap K \cap H$ from $L_2 \cap K \cap H$

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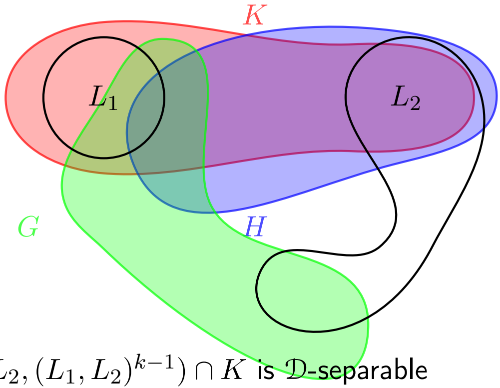

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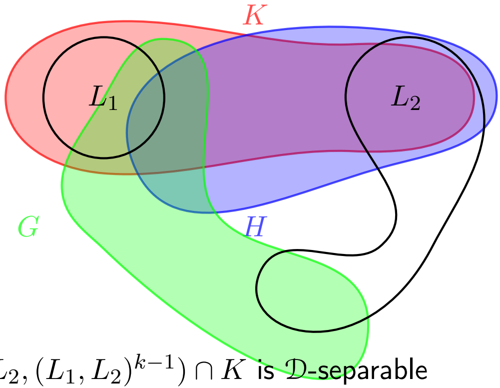

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$(G \cap K) \cup (K \setminus H) \in Bool(\mathcal{D})$ separates L_1 from L_2

$BPol(\mathcal{C})$-separation: Three main steps (2)

Getting rid of Boolean closure

Step 1

Step 2

Step 3

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Tuple $Pol(\mathcal{C})$-separation: Approach (1)

Our input $(L_1, \ldots, L_n)$ is a **tuple of n regular languages**.

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Rule #1 for separation-like problems:

One always looks at several inputs simultaneously.

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Rule $\#$1 for separation-like problems:

One always looks at several inputs simultaneously.

We use a **set of inputs** which has a special structure.

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Given $n \geq 1$, we compute the set $\mathcal{I}^n[\mathbf{L}] \subseteq \mathbf{L}^n$ of all tuples $\overline{L} \in \mathbf{L}^n$ which are **not $Pol(\mathcal{C})$-separable**.

Answer for $(L_1, \ldots, L_n)$ can then be extracted from this information.

Tuple $Pol(\mathcal{C})$-separation: Approach (3)

Given $n \geq 1$, we compute the set $\mathcal{T}^n[\mathbf{L}] \subseteq \mathbf{L}^n$ of all tuples $\overline{L} \in \mathbf{L}^n$ which are **not $Pol(\mathcal{C})$-separable**.

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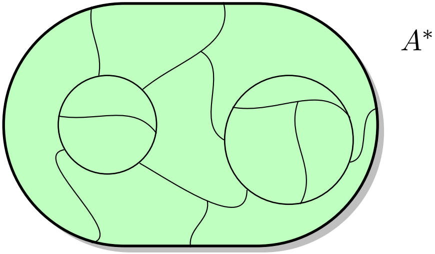
  $\mathcal{C}$ is finite $\Rightarrow$ exists **finest partition** of A^* into languages of $\mathcal{C}$:



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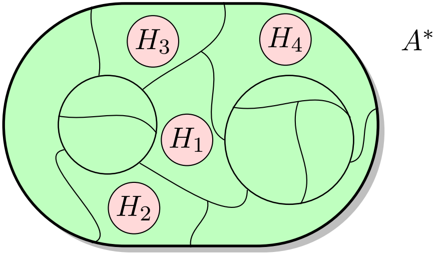

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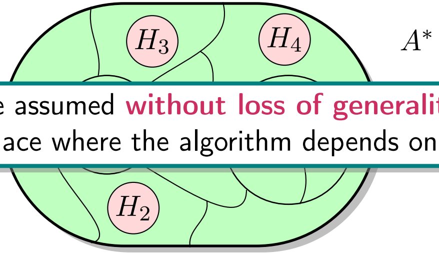

Can be assumed **without loss of generality**
Only place where the algorithm depends on $\mathcal{C}$

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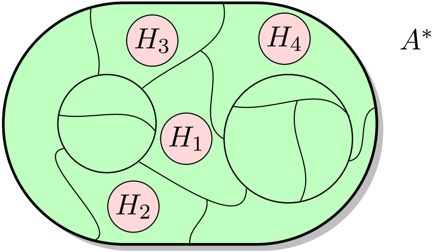
Tuple $Pol(\mathcal{C})$-separation: $\mathcal{C}$-compatibility



Any $H \in \mathbf{L}$ must be **included in a class of this partition**.

We can refine the initial input set to fulfill this condition.

Answer for the original input can be recovered from the refinement.

Least fixpoint computation of $\mathcal{T}^n[\mathbf{L}]$

Given $n \geq 1$, we compute the set $\mathcal{T}^n[\mathbf{L}] \subseteq \mathbf{L}^n$ of all tuples $\overline{L} \in \mathbf{L}^n$ which are **not** $Pol(\mathcal{C})$**-separable**.

$\mathcal{T}^n[\mathbf{L}]$ is computed by **induction on** n:

- $\mathcal{T}^1[\mathbf{L}] \subseteq \mathbf{L}$ is the set of **nonempty** languages in $\mathbf{L}$.

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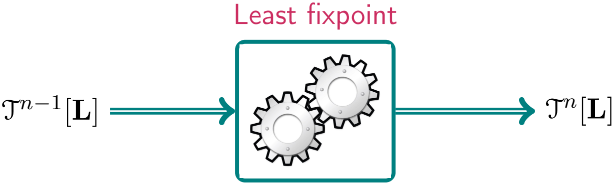

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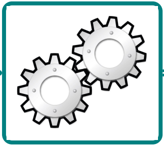
Least fixpoint

$\mathcal{T}^{n-1}[\mathbf{L}] \Longrightarrow$  $\Longrightarrow \mathcal{T}^n[\mathbf{L}]$

  - Computes $\mathcal{T}^n[\mathbf{L}]$ from a subset of trivial tuples
  - **Adds more with operations until a fixpoint is reached**.
  - Requires having $\mathcal{T}^{n-1}[\mathbf{L}]$ in hand.

$BPol(\mathcal{C})$-separation: Three main steps (3)

Step 1

Getting rid of Boolean closure

(L_1, L_2) $BPol(\mathcal{C})$-separable
iff
$\exists k$ $(L_1, L_2)^k$ $Pol(\mathcal{C})$-separable

Step 2

Solving tuple $Pol(\mathcal{C})$-separation:
Input: $(L_1, \ldots, L_n)$ (regular)
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Reuse Step 2 as a subprocedure in our $BPol(\mathcal{C})$ algorithm

$BPol(\mathcal{C})$-separation - General approach

We continue to work with a set of languages $\mathbf{L}$ with appropriate properties.

We compute the set $\mathcal{A}[\mathbf{L}] \subseteq \mathbf{L}^2$ of pairs $(K, L) \in \mathbf{L}^2$ which are **not** $BPol(\mathcal{C})$**-separable**.

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Given two languages L_1, L_2, **TFAE**:

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By the previous steps

Given two languages $K, L \in \mathbf{L}$, **TFAE**:

1. $(K, L) \in \mathcal{A}[\mathbf{L}]$.
2. For all $k \geq 1$, $(K, L)^k$ **not** $Pol(\mathcal{C})$-separable, i.e. $(K, L)^k \in \mathcal{T}^{2k}[\mathbf{L}]$.

$BPol(\mathcal{C})$-separation - Greatest fixpoint

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$$\mathbf{T}_0 = \mathcal{T}^2[\mathbf{L}]$$

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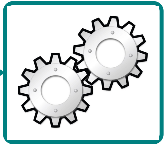

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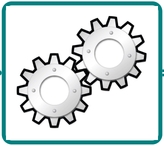

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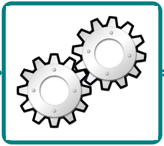

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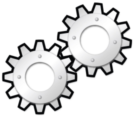
$\cup|$

$\mathbf{T}_1$

Least fixpoint



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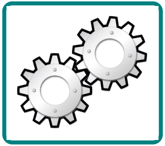

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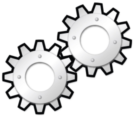
$\supseteq$

$\mathbf{T}_1$

$\supseteq$

$\mathbf{T}_2$

Least fixpoint



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$\cup|$

$\mathbf{T}_1$

$\cup|$

$\mathbf{T}_2$

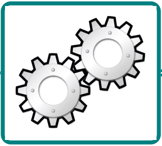
$\vdots$

$\cup|$

$\vdots$

$\mathbf{T}_n$

Least fixpoint



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Eventually, a **fixpoint is reached**, $\mathbf{T}_n = \mathbf{T}_{n+1} = \mathcal{A}[\mathbf{L}]$

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The $BPol(\mathcal{C})$-algorithm is a **greatest fixpoint** whose **operation is a least fixpoint**

$\mathbf{R} \subseteq \mathbf{L}^3$

$\cup|$

$\mathbf{T}_2$

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Conclusion

Conclusion (1)

Everything we know is captured by only **four generic results**:

1. $\mathcal{C}$ finite $\Rightarrow$ $Pol(\mathcal{C})$-separation decidable.
2. $\mathcal{C}$ finite $\Rightarrow$ $BPol(\mathcal{C})$-separation decidable.
3. $\mathcal{C}$ finite $\Rightarrow$ $Pol(BPol(\mathcal{C}))$-separation decidable.
4. For any $\mathcal{C}$,
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Bonus corresponding to enrichment with successor

For any $\mathcal{C}$, $\mathcal{C}$-separation decidable $\Rightarrow$ $\mathcal{C}^+$-separation decidable.

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Some words about **complexity**:

1. Complexity depends on $|\mathcal{C}|$ (tied to the implicit alphabet).
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1. Complexity depends on $|\mathcal{C}|$ (tied to the implicit alphabet).
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Conclusion (2)

Future work

We have the result,

- For any $\mathcal{C}$,
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A nontrivial corollary of the $BPol(\mathcal{C})$-algorithm is as follows:

- For any $\mathcal{C}$,
  $\mathcal{C}$-"something" decidable $\Rightarrow$ $BPol(\mathcal{C})$-membership decidable.

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Future work

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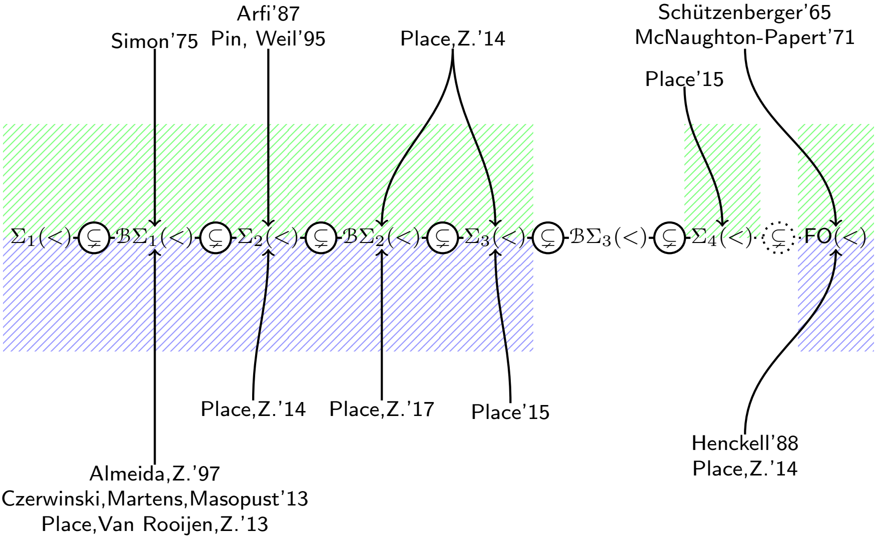
$\mathcal{B}\Sigma_2(<)$-"something" seems to be decidable which would yield a membership algorithm for $\mathcal{B}\Sigma_3(<)$.


Arfi'87
Pin, Weil'95
Simon'75
Place,Z.'14
Schützenberger'65
McNaughton-Papert'71
Place'15
$\Sigma_1(<) \subsetneq \mathcal{B}\Sigma_1(<) \subsetneq \Sigma_2(<) \subsetneq \mathcal{B}\Sigma_2(<) \subsetneq \Sigma_3(<) \subsetneq \mathcal{B}\Sigma_3(<) \subsetneq \Sigma_4(<) \subsetneq$ FO$(<)$
Place,Z.'14
Place,Z.'17
Place'15
Henckell'88
Place,Z.'14
Almeida,Z.'97
Czerwinski,Martens,Masopust'13
Place,Van Rooijen,Z.'13

Thank You