Separation and concatenation hierarchies (Part I)

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Joint work with Marc Zeitoun

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Investigating Logics over Words

Main Objective

Descriptive Formalisms

First-Order Logic $\mathbf{FO}(<)$
or Fragments such as:
$\mathrm{FO}(+1), \Sigma_i, \mathcal{B}\Sigma_i$
2-Variable FO: $\mathbf{FO}^2(<)$
$\ldots$

Words
acbacbca

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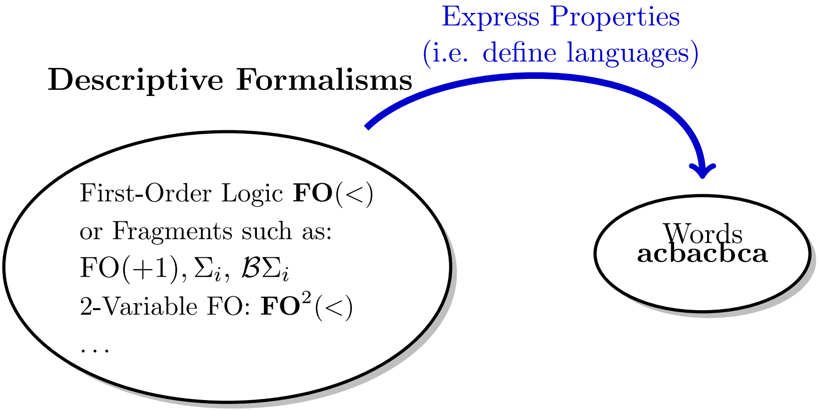

Main Objective

Express Properties
(i.e. define languages)

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$\dots$

Words
acbacbca

Objective: For each fragment, **understand** what it can express.

i.e. **What languages belong to the associated class $\mathcal{C}$?**

First-Order Logic over Words (FO(<))

$$\begin{array}{cccccccccccc} a & b & b & b & c & a & a & a & c & a & \in A^* \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \end{array}$$

- A word is a sequence of labeled positions.
- Positions can be quantified.

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$$\forall x\ (a(x) \Rightarrow \exists y\ (b(y) \wedge (y < x)))$$

"for any a in the word, there is a b to its left"

Each sentence defines a language

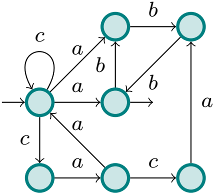
$\Rightarrow$ FO(<) defines a **class of languages**.

We want to **understand classes of languages** (defined by logic)

Methodology: The membership problem

Given such a class $\mathcal{C}$, the goal is to solve the associated **membership problem**:

L a regular language



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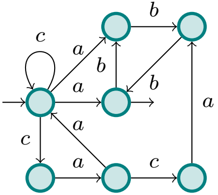

Does L belong to the class $\mathcal{C}$?

There are two stages to the problem:

- Stage 1: get an **algorithm** that decides it.
- Stage 2: find a generic way to **construct a sentence** witnessing membership of L in $\mathcal{C}$ when it exists.

Example - McNaughton-Papert-Schützenberger

Given a regular language L, the following properties are equivalent:

- L is definable in FO($<$)
- The minimal automaton of L is counter-free
- The syntactic monoid of L is aperiodic

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Why is it interesting ?

1. The theorem itself is an effective description of the class FO($<$).
2. The proofs are constructive: if we have the minimal automaton in hand, we can **construct a sentence** for L by induction.
   $\Rightarrow$ We get normal forms for FO($<$) sentences over words.

Altogether, we learn a lot about FO($<$) from this theorem

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- Understanding a class $\mathcal{C}$ = solving $\mathcal{C}$-**membership**.
- Proof provides a **canonical representation** of languages in $\mathcal{C}$.

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  - For other structures: infinite words, finite trees.

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- **Successful methodology since the 70s**, reproduced
  - For other logical classes on words (eg, several restrictions of FO).
  - For other structures: infinite words, finite trees.

- Still, the methodology **fails for important classes**.

The big problem: quantifier alternation

Quantifer Alternation: Classifying Sentences

A simple sentence:

$$\exists x \exists y \forall z \; b(x) \wedge b(y) \wedge ((x < z < y) \Rightarrow a(z)))$$

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Complicated = High Quantifier Alternation

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Level n: $\Sigma_n(<)$ sentence (in prenex normal form)

$$\underbrace{\exists x_1, \ldots, x_{n_1} \forall y_1, \ldots, y_{n_2} \cdots\cdots}_{n \text{ blocks (starting with } \exists)} \; \underbrace{\varphi(\bar{x}, \bar{y}, \ldots)}_{\text{quantifier-free}}$$

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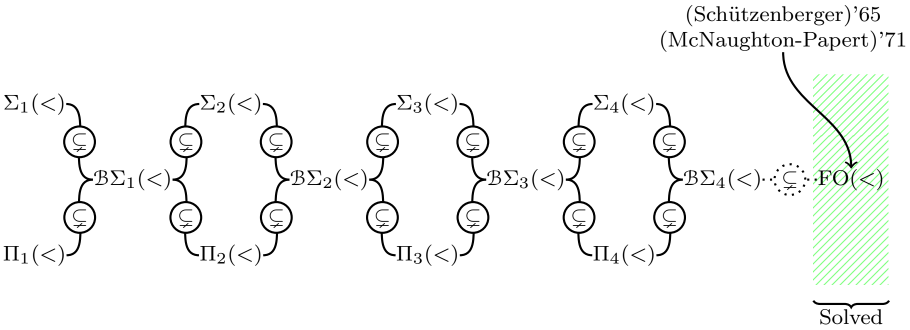
$\Sigma_n(<)$ not closed under complement $\Rightarrow \mathcal{B}\Sigma_n(<)$

$\mathcal{B}\Sigma_n(<)$ sentence $=$ **Boolean combination** of $\Sigma_n(<)$ sentences.

Quantifier Alternation: Membership state of the art



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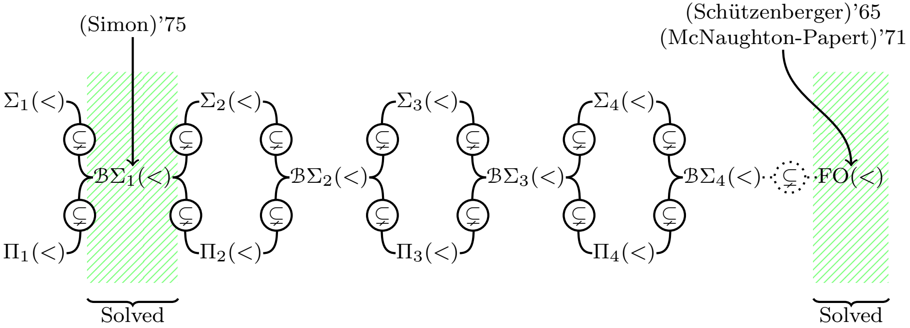

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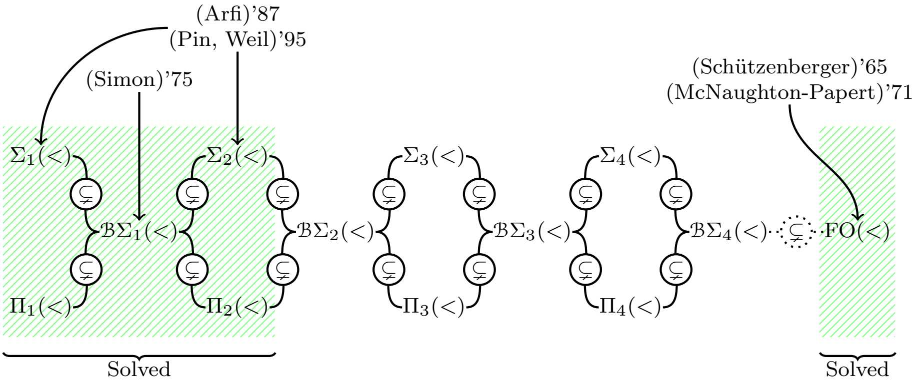

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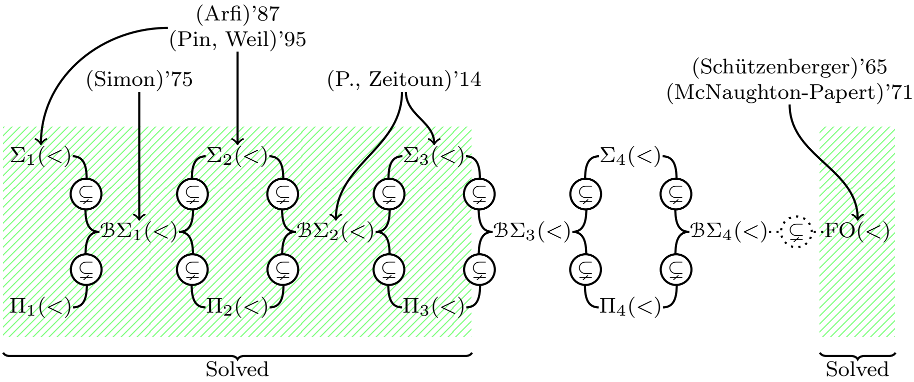

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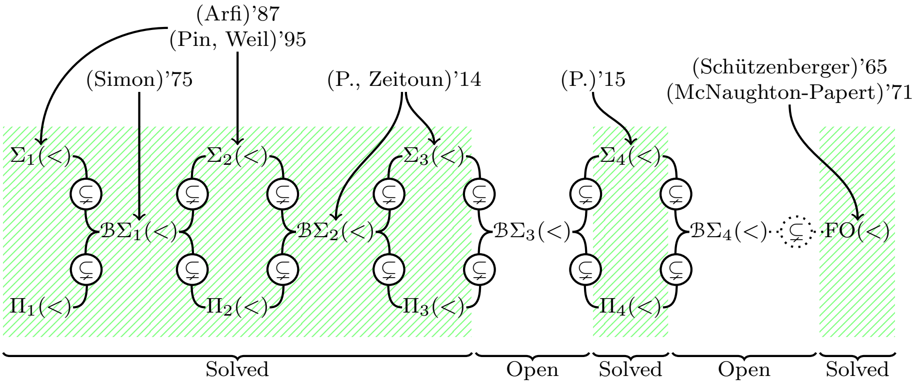

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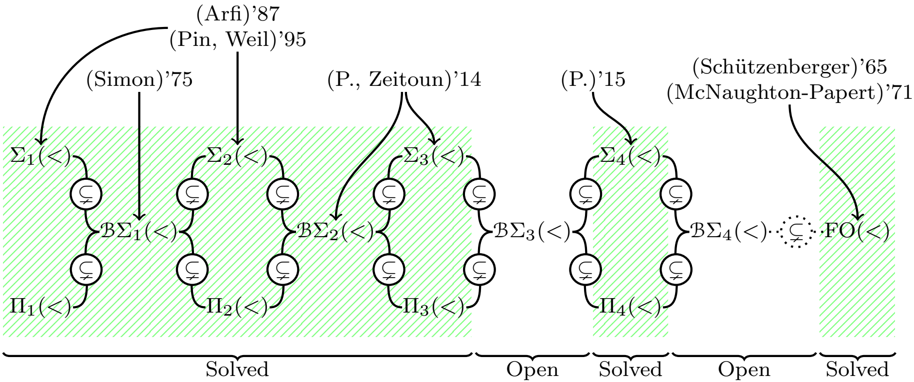

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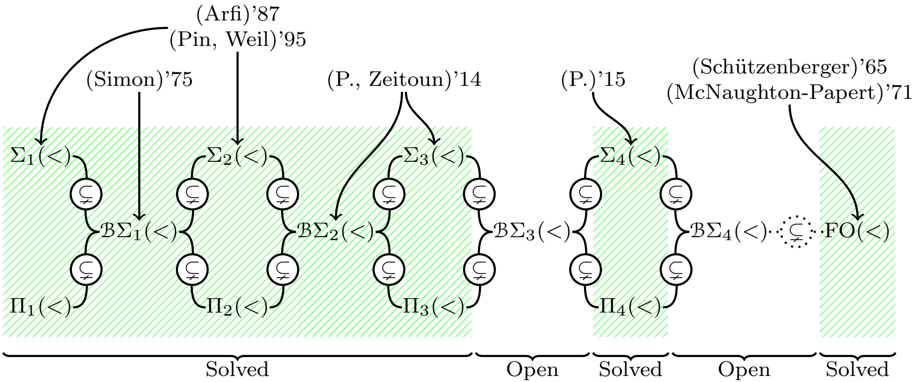


How are this results obtained ?

The previous slides only present a **third of the story** (at best).

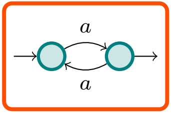
The Separation Problem

Definition

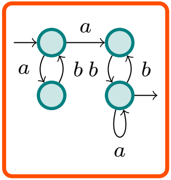
Given a class of languages $\mathcal{C}$ (for example a level in the hierarchy), decide the following problem:

L_1, L_2 two regular languages

L_1



L_2



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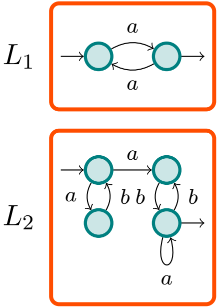
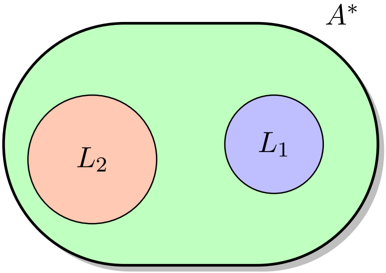


Can L_1 be separated from L_2 with a language of $\mathcal{C}$?



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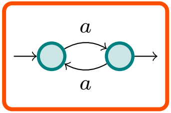
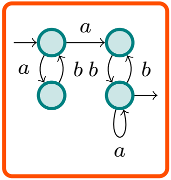


L_2



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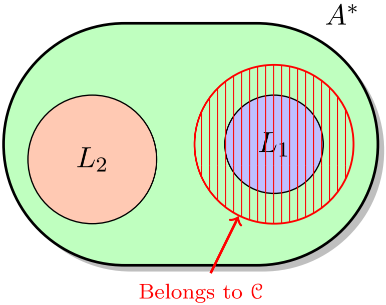

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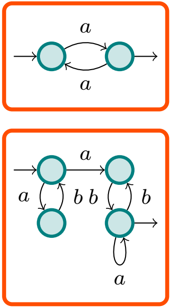

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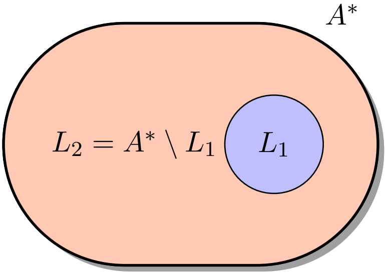

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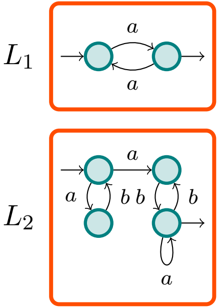

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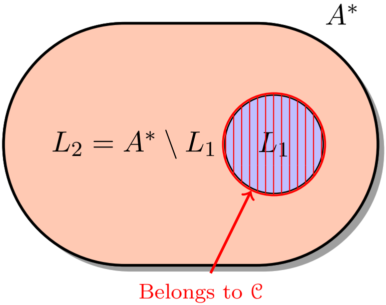

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Motivation for Separation (1)

Negative aspect:

- Usually harder than membership.

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- More rewarding with respect to the investigated class.

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Negative aspect:

- ☹ Usually harder than membership.

Positive aspects:

- ☺ More rewarding with respect to the investigated class.
- ☺ Membership for $\mathcal{C}$ = Techniques applying to languages in $\mathcal{C}$ only.
  Separation for $\mathcal{C}$ = Techniques applying to **all languages**.

Membership for $\mathcal{C}$

Given a language L:

1. Does $L \in \mathcal{C}$?
2. If so, compute a description of L in $\mathcal{C}$.

Separation for $\mathcal{C}$

Given two languages L_1, L_2:

1. Can we **approximate** L_1 with some $K \in \mathcal{C}$? (allowed approximations given by L_2)
2. If so, compute $K \in \mathcal{C}$ realizing this approximation.

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All results that we have today for the hierarchy are **based on separation** (or more general problems):

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Transfer theorem (P.,Zeitoun)'14

For all $n \geq 1$,

$$\Sigma_n\text{-separation decidable} \Rightarrow \Sigma_{n+1}\text{-membership decidable}$$

Important Remark

Separation is **harder** than membership. The above above does **not** solve the whole hierarchy.

Transfer theorem: Σ_{n-1}-separation $\Rightarrow$ Σ_n-membership

Notation, for two states p, q: $L_{p,q} = \{w \mid p \xrightarrow{w} q\}$

Forbidden Patterns and Separation

A regular language **is definable in $\Sigma_\mathbf{n}$** iff its minimal automaton has **no pattern**:

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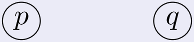

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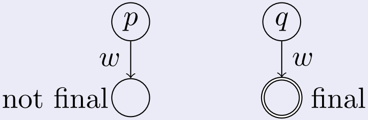

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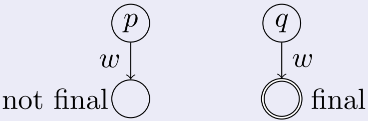

where $L_{p,q}$ is **not Σ_{n-1}-separable** from $L_{p,p} \cap L_{q,q}$

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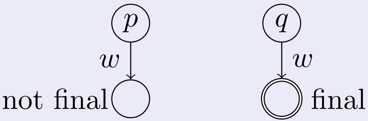

where $L_{p,q}$ is **not $\Sigma_{\mathbf{n-1}}$-separable** from $L_{p,p} \cap L_{q,q}$

Corollary

Solving **$\Sigma_{\mathbf{n-1}}$-separation** yields a solution for **$\Sigma_{\mathbf{n}}$-membership**.

Limits of this approach

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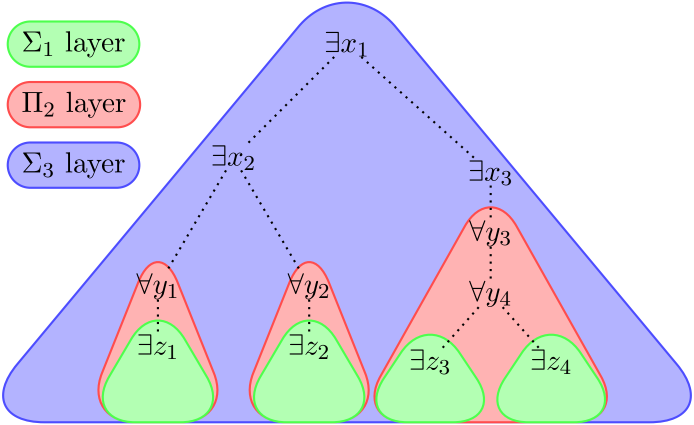
Let us explain why.

Hard part for both membership and separation:
Generic construction of **descriptions in $\mathcal{C}$**.

This is also the case for the transfer theorem.

Construction of Σ_n sentences

A Σ_n sentence is **layered**: Consider a Σ_3 sentence



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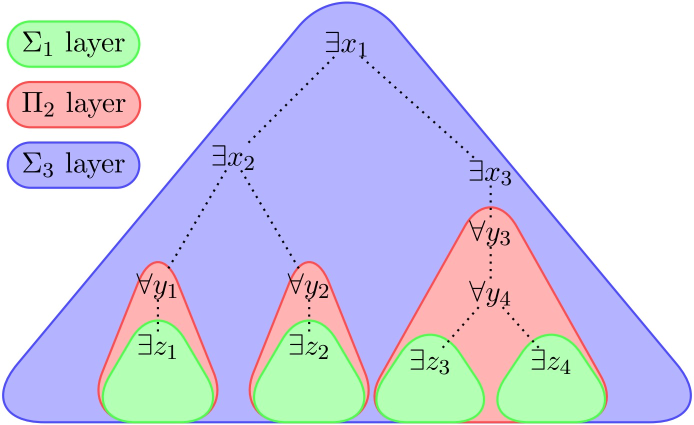

A generic construction should have several phases:
one for each layer

Construction of Σ_n sentences in the transfer theorem

Starting from a **DFA** $\mathcal{A}$ satisfying the transfer theorem, one builds a Σ_n sentence as follows:

- All languages needed for the **layers below Σ_{n-1}** are Σ_{n-1}-separators of $L_{p,q}$ from $L_{p,p} \cap L_{q,q}$ for some states p, q of $\mathcal{A}$.
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- The lower layers are built by **approximating these languages** with Σ_{n-1}-separation.

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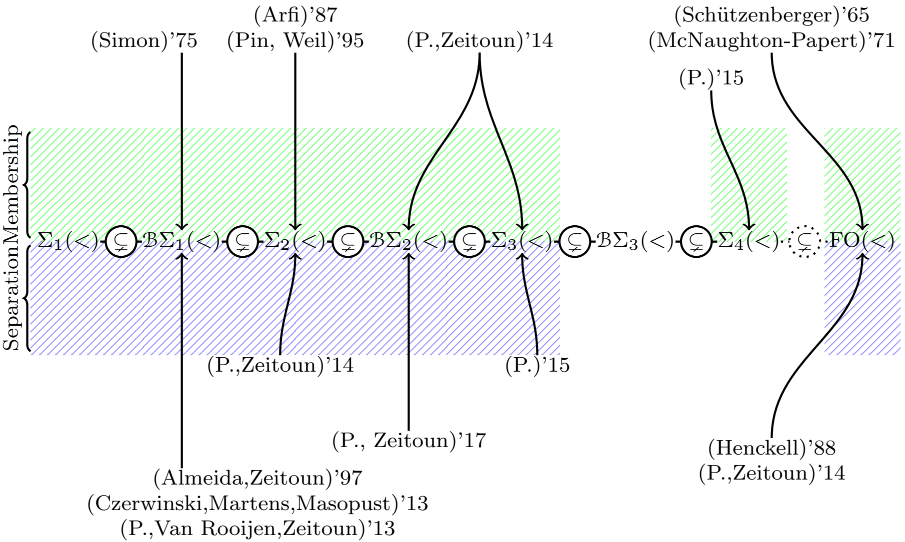
Separation is different: we do not have the Σ_n-layer in hand.
$\Rightarrow$ All layers must be considered simultaneously.

Current state of the art: Separation



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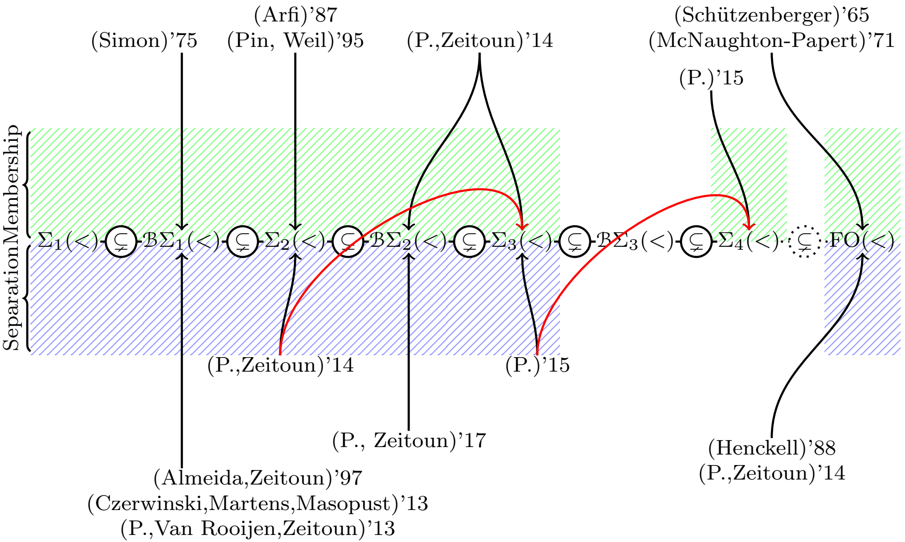

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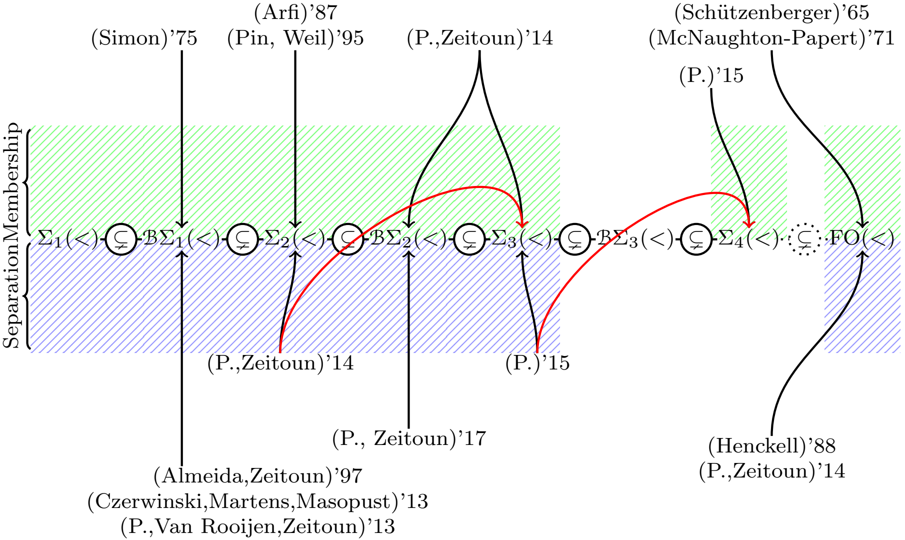

We are still missing **one third of the story**.

Concatenation hierarchies

Star-free languages (1)

McNaughton-Papert'71

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Star-free languages

- $\emptyset$ and A^* are star-free.
  $\Rightarrow$ Corresponds to the FO($<$) sentences $\perp$ and $\top$.
- Closed under **union**, **union** and **complement**.
  $\Rightarrow$ Corresponds to Boolean connectives in FO($<$).

Star-free languages (1)

McNaughton-Papert'71

Given a regular language L, the following properties are equivalent:

- L may be defined by an FO($<$) sentence.
- L is **star-free**.

Star-free languages

- $\emptyset$ and A^* are star-free.
  $\Rightarrow$ Corresponds to the FO($<$) sentences $\perp$ and $\top$.
- Closed under **union**, **union** and **complement**.
  $\Rightarrow$ Corresponds to Boolean connectives in FO($<$).
- Closed under **marked concatenation**:

$$\text{Given } a \in A \quad K, L, a \mapsto KaL$$

  $\Rightarrow$ Corresponds to existential quantification in FO($<$).

$$\exists x \; a(x) \wedge \varphi_K^{<x}(x) \wedge \varphi_L^{>x}(x)$$

Star-free languages (2)

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However, in generic constructions of FO($<$) sentences, this additional **freedom is never used**.

For building FO($<$) languages, one always starts from $\emptyset$ and A^* using only **Boolean operations** and **marked concatenations**.

This is also the case for classes in the quantifier alternation hierarchy of FO($<$).

The Straubing Thérien Hierarchy'81

Classifies the star-free languages into half and full levels:

$$0 \longrightarrow \tfrac{1}{2} \longrightarrow 1 \longrightarrow \tfrac{3}{2} \longrightarrow 2 \longrightarrow \tfrac{5}{2} \longrightarrow 3 \cdots\cdots$$

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Polynomial closure

$Pol(\mathcal{C})$ built by closing the class $\mathcal{C}$ under:

- Union ($\bigcup$).
- Intersection ($\bigcap$).
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((Thomas)'82 (Perrin,Pin)'86)

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Generic template: Concatenation hierarchies

Previous slide is an example of a **generic** construction.

Generic template: Concatenation hierarchies

0
Basis:
class $\mathcal{C}$

$\mathcal{C}$ must be closed under:

- **Boolean operations**.
- **Quotients**. For $L \in \mathcal{C}$, $w \in A^*$,

$$w^{-1}L \stackrel{\text{def}}{=} \{u \in A^* \mid wu \in L\} \in \mathcal{C}$$

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Generic template: Concatenation hierarchies

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$$0 \xrightarrow{\text{Pol}} \tfrac{1}{2} \xrightarrow[\text{Bool}]{} 1$$

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All results for quantifier alternation **can be lifted as generic results** for concatenation hierarchies whose basis is finite.

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All results for quantifier alternation **can be lifted as generic results** for concatenation hierarchies whose basis is finite.

Before we explain how, let us further motivate the introduction of concatenation hierarchies.

The logical connection is generic (1)

Given a basis $\mathcal{C}$, we define a **variant FO($\mathcal{C}$) of first-order logic** equipped with the following signature:

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The concatenation hierarchy of basis $\mathcal{C}$ corresponds to the quantifier alternation hierarchy within $\mathrm{FO}(\mathcal{C})$.

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\end{tikzcd}$$

The logical connection is generic: Examples

The Straubing-Thérien hierarchy

Basis $\mathcal{C} = \{\emptyset, A^*\} \Rightarrow \mathrm{FO}(<)$

- $I_{A^*}(x, y)$ is $x < y$.
- $P_{A^*}(x), S_{A^*}(x), N_{A^*}$ are equivalent to $\top$.
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The dot-depth hierarchy (Brzozowski,Cohen)'71

Basis $\mathcal{C} = \{\emptyset, \{\varepsilon\}, A^+, A^*\} \Rightarrow \mathrm{FO}(<, +1, min, max, \varepsilon)$ (Thomas)'82

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Generic separation results

Generic separation results (1)

$$\underset{\substack{\text{Basis:}\\ \text{class } \mathcal{C}}}{0} \xrightarrow{\text{Pol}} \tfrac{1}{2} \xrightarrow[\text{Bool}]{} 1 \xrightarrow{\text{Pol}} \tfrac{3}{2} \xrightarrow[\text{Bool}]{} 2 \xrightarrow{\text{Pol}} \tfrac{5}{2} \xrightarrow[\text{Bool}]{} 3 \cdots\cdots$$

Generic Separation Results

If $\mathcal{C}$ is **finite**, then **separation is decidable** for the following,

Generic separation results (1)

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If $\mathcal{C}$ is **finite**, then **separation is decidable** for the following,

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2. $BPol(\mathcal{C})$ (i.e. $Bool(Pol(\mathcal{C}))$).

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3. $Pol(BPol(\mathcal{C}))$.

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Basis:
class $\mathcal{C}$

Advantages:

- 🙂 These results treat many classes.
- 🙂 **All we know** about separation is captured by **three results**.
- 🙂 We **pinpoint the hypotheses** which we really need.

Generic separation results (2)

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Wait ! The speaker lied ! Refund my 53 €!!!!!!

The results for FO($<$) went **one full level higher**, didn't they ?

The almighty alphabet argument

Logical point of view (hierarchy within FO($<$)):

$$\mathcal{B}\Sigma_0(<) - \Sigma_1(<) - \mathcal{B}\Sigma_1(<) - \Sigma_2(<) - \mathcal{B}\Sigma_2(<) - \Sigma_3(<) - \mathcal{B}\Sigma_3(<)$$

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Basis:
$\{\emptyset, A^*\}$

Finite class AT
(**Alphabet testable**)
Boolean combinations of languages of the form A^*aA^* for some $a \in A$

The almighty alphabet argument

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Summary

Everything we know is captured by only **four generic results**:

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Additional important result:

5. Generic reduction. For any half or full level n:

Transfer of separation

Level n in the dot-depth reduces to level n in the Straubing-Thérien.

Conclusion

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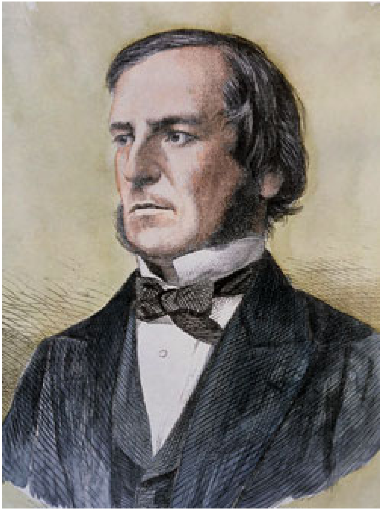
Some words about **complexity**:

1. Complexity depends on $|\mathcal{C}|$ (tied to the implicit alphabet).
2. $Pol(\mathrm{AT})$ and $BPol(\mathrm{AT})$ are **PSpace**(-complete).
3. If the alphabet is fixed, or $|\mathcal{C}|$ is constant,

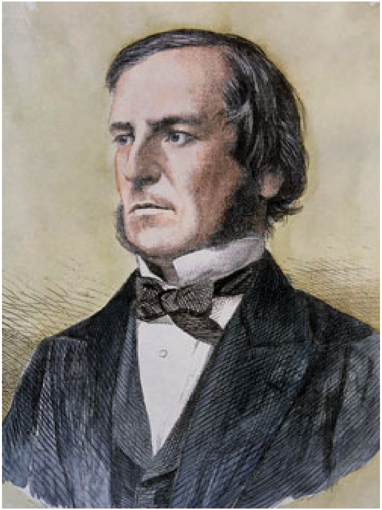
   $Pol(\mathcal{C})$-separation and $BPol(\mathcal{C})$-separation are in **PTime**

What you should know



George Boole
(1815-1864)

What you should know



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This guy is evil

We don't understand negation

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Only three results

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2. $\mathcal{C}$ finite $\Rightarrow$ $BPol(\mathcal{C})$-separation decidable.
3. $\mathcal{C}$ finite $\Rightarrow$ $Pol(BPol(\mathcal{C}))$-separation decidable.

We are only able to handle **one** negation.

We don't understand negation (2)

Let's consider two other examples

Two-variables first-order logic ($FO^2(<)$): plenty of negation

Separation is decidable. Operations used to build separators:

- Union.
- Concatenations.

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First-order logic ($FO(<)$): even more negation

Separation still decidable. Operations used to build separators:

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- Kleene star (simulated with negation in special situations)

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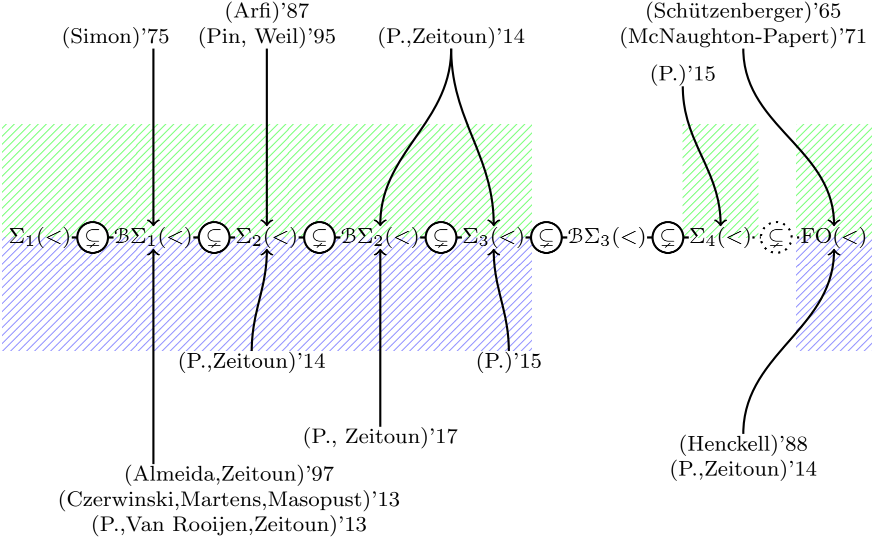
There are **three** operations that we understand: union, concatenation and (to a lesser extent) Kleene star. Complement is evil.



(Arfi)'87
(Simon)'75
(Pin, Weil)'95
(P.,Zeitoun)'14
(Schützenberger)'65
(McNaughton-Papert)'71
(P.)'15
$\Sigma_1(<) \subsetneq \mathcal{B}\Sigma_1(<) \subsetneq \Sigma_2(<) \subsetneq \mathcal{B}\Sigma_2(<) \subsetneq \Sigma_3(<) \subsetneq \mathcal{B}\Sigma_3(<) \subsetneq \Sigma_4(<) \subsetneq FO(<)$
(P.,Zeitoun)'14
(P.)'15
(P., Zeitoun)'17
(Henckell)'88
(P.,Zeitoun)'14
(Almeida,Zeitoun)'97
(Czerwinski,Martens,Masopust)'13
(P.,Van Rooijen,Zeitoun)'13

Thank You