Separation of tropical automata

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Separation workshop

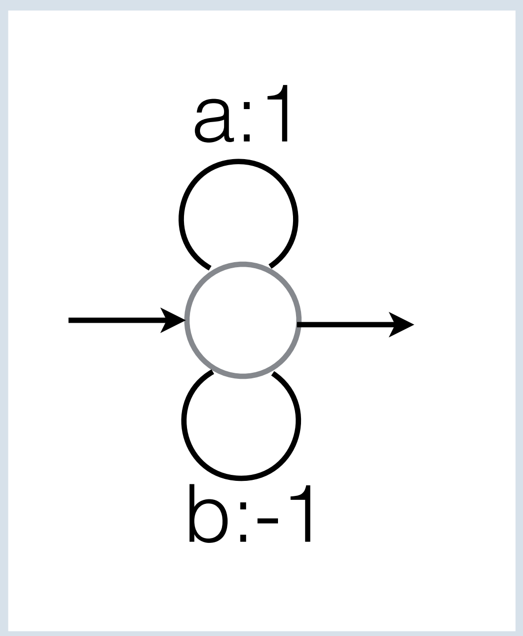
joint work with Sylvain Lombardy

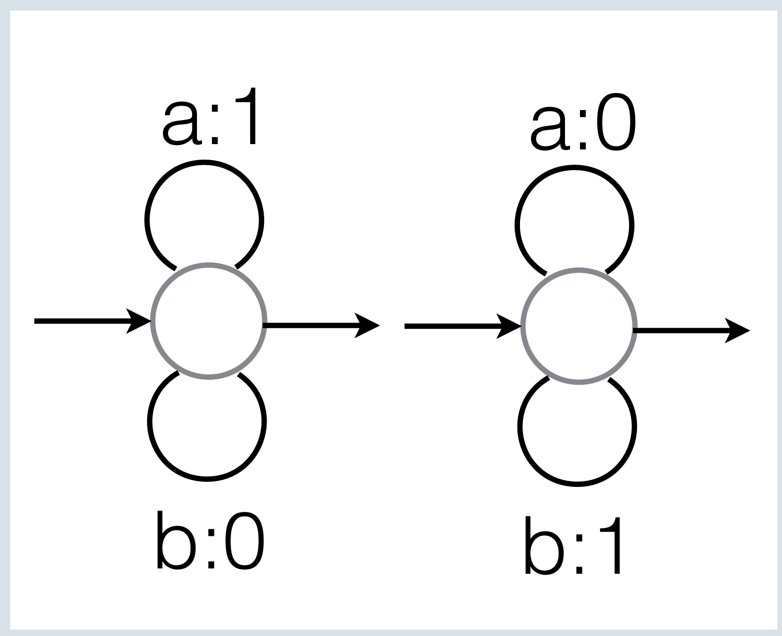
Warsaw, July 14, 2017

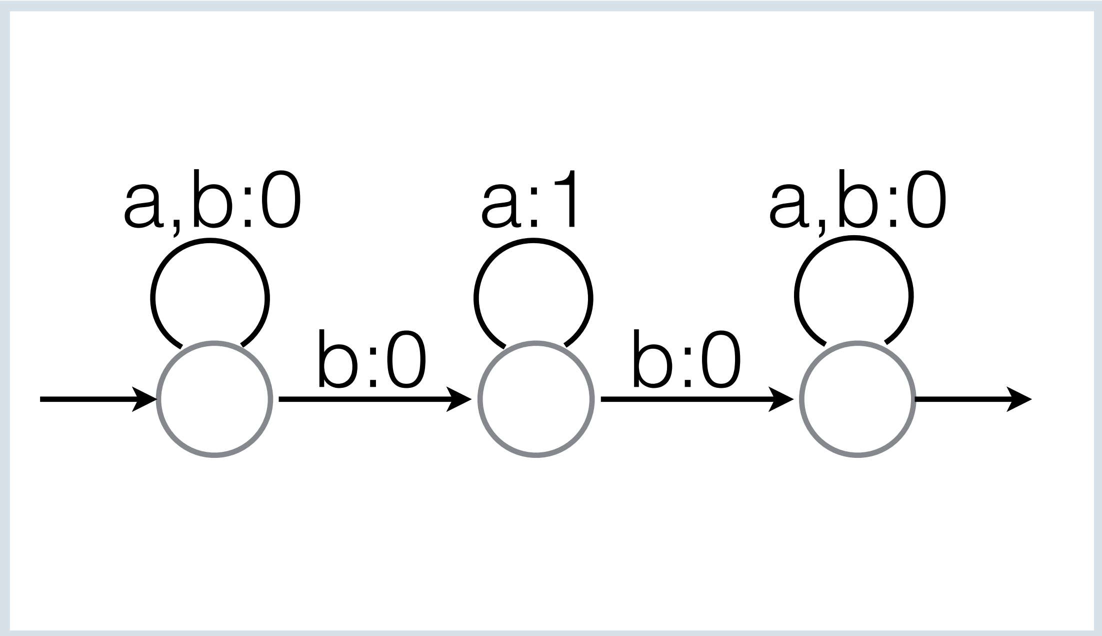
Unambiguous tropical automata

Tropical automata

Hashiguchi
Simon







A tropical automaton is a non-deterministic automaton weighted by integers. It computes a function from words to integers (and +/-∞).

min-+: outputs the minimum over all accepting runs of the total weight

max-+: outputs the maximum over all accepting runs of the total weight

Theorem: It is decidable if a min-+ rational function f satisfies f≥0.
(resp. g≤0 for g max-+)

Theorem [Krob94]: It is undecidable if a max-+ rational function f satisfies f≥0. (resp. g≤0 for g min-+)

Note that min-+ and max-+ semantics coincide over unambiguous automata. This yields unambiguous tropical automata.

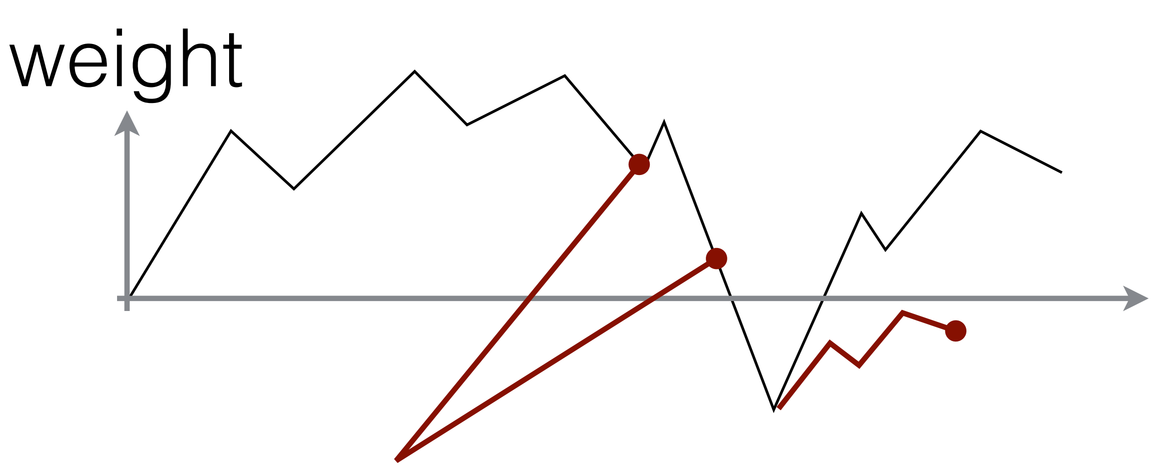
Being both min and max

Theorem [Lombardy&Mairesse06]: A function from words to integers that is both min-+ and max-+ rational is (effectively) recognized by an unambiguous tropical automaton.

Lemma A: Given a min-+ function f such that f≥0, the set of accepting runs of weight 0 is (effectively) regular.

No accepting run has a negative weight.

Proof: *Consider an initial run, and draw:*



Claim 2: *There is k' such that the partial weight cannot decrease of more than k'.*

Claim 1: *There exists k such that the run never goes below -k.*

Thus, all runs of weight 0 have all their intermediate values in the interval [-k,k'].

Hence, an automaton keeping weights in this interval can recognize runs of weight 0.

Being both min and max

Theorem [Lombardy&Mairesse06]: A function from words to integers that is both min-+ and max-+ rational is (effectively) recognized by an unambiguous tropical automaton.

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(

Along the same ideas:

Proposition [Krob94] **(Fatou property):**
If a min-+ rational function f is such that f≥0
Then it is recognized by a min-+ automaton with only non-negative weights.

)

Being both min and max

Theorem [Lombardy&Mairesse06]: A function from words to integers that is both min-+ and max-+ rational is (effectively) recognized by an unambiguous tropical automaton.

Lemma A: Given a min-+ function f such that f≥0, the set of accepting runs of weight 0 is (effectively) regular.

Proof of the theorem: *Consider a min-+ automaton A for f, and a max-+ automaton B for g such that f=g.*

Construct the product automaton of A and B, with two weights
(a) the A weights, and
(b) the A-B weights.

The min-+ (b)-automaton computes f-g=0.

By Lemma A, we can restrict it to the runs of (b)-weight 0.

The resulting (a) automaton is such that:

a) all inputs u have an accepting run
b) all accepting runs have the weight f(u)

It can be made unambiguous by keeping the lexicographic least run.

Our result

Theorem [Lombardy&Mairesse06]: A function from words to integers that is both min-+ and max-+ recognized is (effectively) recognized by an unambiguous tropical automaton.

N or Z or Q is the same

Theorem: Given a max-+ regular function f and a min-+ regular function g such that $f \leq g$, then there exists an unambiguous regular function h such that

$$f \leq h \leq g .$$

Remark: It generalizes [Lombardy&Mairesse06]

We shall give two quite different proofs, first over for Z, then over R.

Separation of integer-weighted tropical automata

Tool 1: regular lookahead

Informally: an unambiguous automaton is a deterministic automaton equipped with a regular lookahead.

In particular in our case, a regular lookahead can give access to the following information:
- for all states p of an automaton (non-deterministic), is it at the origin of an accepting run on the rest of the word ?

From now: all states of an automaton that we mention are supposed to be at the origin of a run on the current word. This is the only form of lookahead that is needed; all the rest is deterministic.

Tool 2: witness of inequality

A witness of the inequality is a map:

$$d \colon Q_{\mathtt{Max}} \times Q_{\mathtt{Min}} \to \mathbb{R} \cup \{\perp\}$$

defined only if the states may coexist: if there exists a word such that both automata assume these states at the same position.

such that:

- for all $(p, a, x, p') \in \mathtt{Max}$ and $(q, a, y, q') \in \mathtt{Min}$,

$$d(p,q) + x \leqslant d(p',q') + y$$

- for p, q initial states, $d(p,q) \geqslant 0$
- for p, q final states, $d(p,q) \leqslant 0$

Lemma: $[\![\mathtt{Max}]\!] \leqslant [\![\mathtt{Min}]\!]$ if and only if there exists a witness of inequality.

Proof left to right: Define $d(p,q) = \inf_{v \in A^*, (p,q) \to^u F} ([\![\mathtt{Min}_q]\!](v) - [\![\mathtt{Max}_p]\!](v))$

Proof right to left: sum the terms.

Remark: Whenever $p \to^u F_{\mathtt{Max}}$, $q \to^u F_{\mathtt{Min}}$ and $y - x \geqslant d(p,q)$,

$$[\![\mathtt{Max}_p(x)]\!](u) \leqslant [\![\mathtt{Min}_q(y)]\!](u)$$

Construction for integers

The idea is to determinize the **Max** automaton:

- keeping track of all maximum runs of the Max automaton reaching each state.
- no run is ever lost (thanks to the lookahead)
- one keeps only the maximum value in the value
- the other values are obtained by keeping the differences in the state.

configurations reachable after reading a word u

(p,30)
(q,2)
(r,35)
~~(s,8)~~

one configuration of the new automaton

$$\left(\begin{vmatrix} p & -5 \\ q & -33 \\ r & -0 \end{vmatrix}, 35 \right)$$

the maximal value is kept among all states

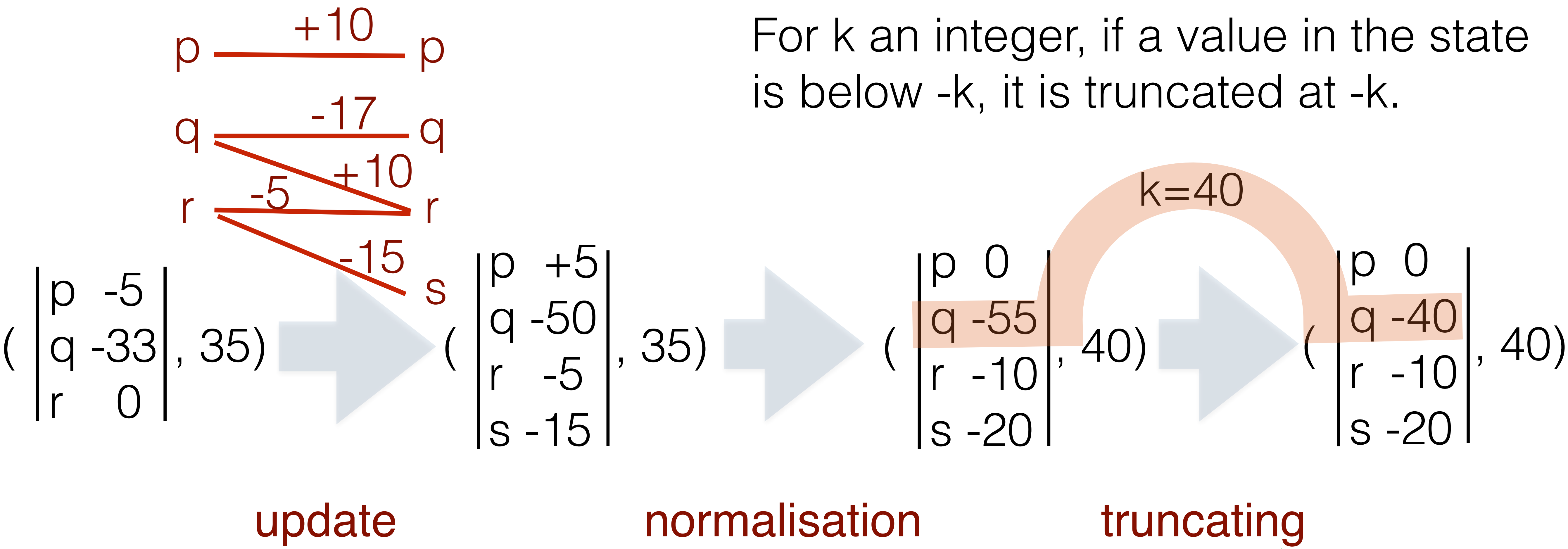
the difference is kept in the state

One keeps only the states that yield to an accepting run

This is an infinite deterministic automaton with lookahead for the same function.

k over approximant

For k an integer, if a value in the state is below -k, it is truncated at -k.

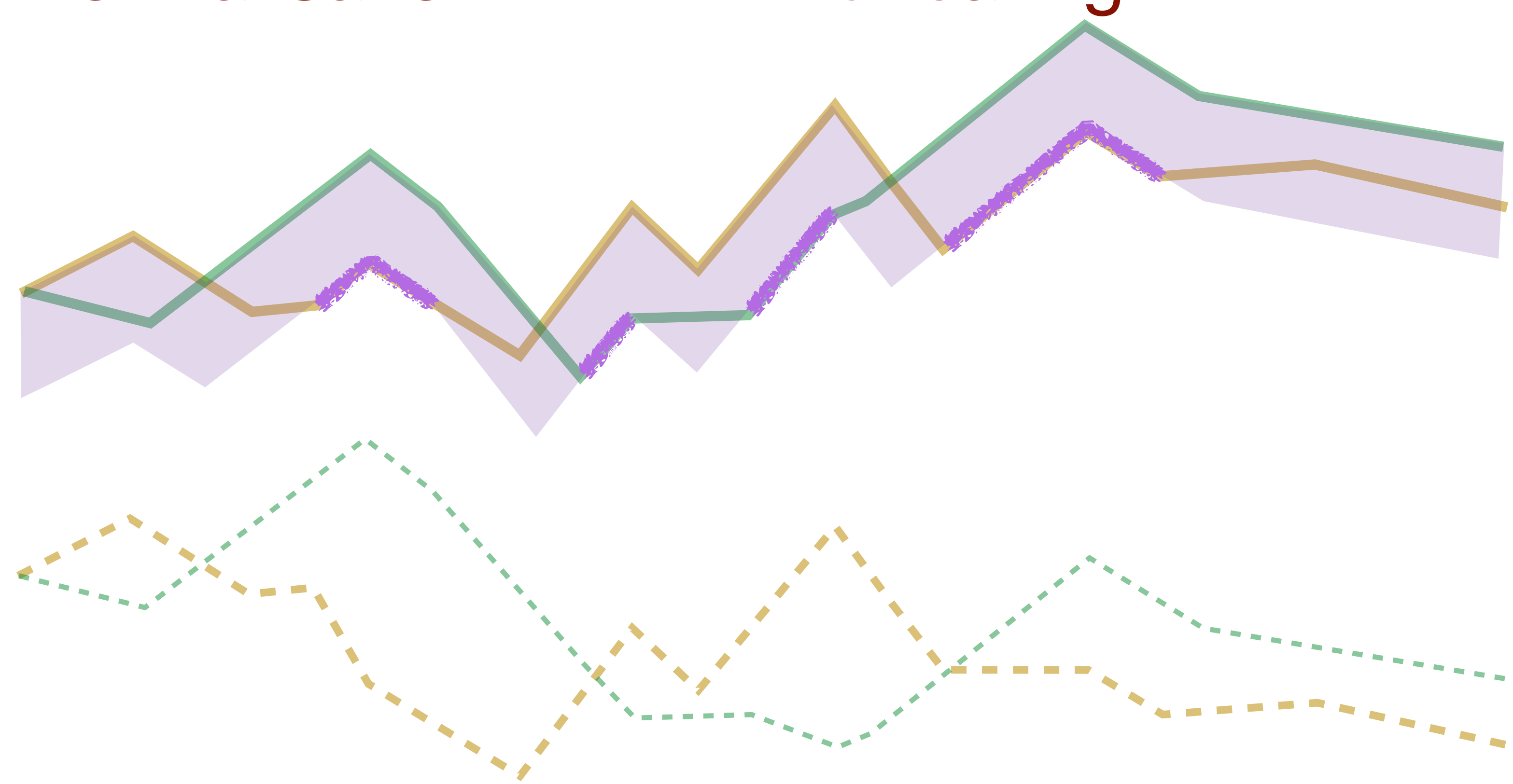


1) $\mathtt{MaxApprox}_k$ is finite

2) $[\![\mathtt{Max}]\!] \leqslant [\![\mathtt{MaxApprox}_k]\!]$

3) for $k = 2 \max_{p,q} |d(p,q)|$

$$[\![\mathtt{MaxApprox}_k]\!] \leqslant [\![\mathtt{Min}]\!]$$



Separation of real-weighted tropical automata

The real case

Theorem: Given a max-+ regular function f and a min-+ regular function g such that $f \leq g$, then there exists an unambiguous regular function h such that

$$f \leq h \leq g \,.$$

Remark: the previous proof does not work. Indeed, keeping differences in [-k,0] does not yield finitely many states of the unambiguous automaton.

We keep the lookahead technique:
every state belongs to an accepting run.

The monoids of matrices

Max-+ matrix monoid $M_{\texttt{Max}} = (\mathbb{R} \cup \{-\infty\})^{Q_{\texttt{Max}} \times Q_{\texttt{Max}}}$ in $(\mathbb{R} \cup \{-\infty\}, -\infty, 0, \max, +)$

Min-+ matrix monoid $M_{\texttt{Min}} = (\mathbb{R} \cup \{+\infty\})^{Q_{\texttt{Min}} \times Q_{\texttt{Min}}}$ in $(\mathbb{R} \cup \{+\infty\}, +\infty, 0, \min, +)$

Two monoid morphisms

$$\rho_{\texttt{Max}} : \Sigma^* \longrightarrow M_{\texttt{Max}}$$
$$\rho_{\texttt{Min}} : \Sigma^* \longrightarrow M_{\texttt{Min}}$$

Well known fact:

$$[\![\texttt{Max}]\!](u) = I^t_{\texttt{Max}} \cdot \rho_{\texttt{Max}}(u) \cdot F_{\texttt{Max}}$$
$$[\![\texttt{Min}]\!](u) = I^t_{\texttt{Min}} \cdot \rho_{\texttt{Min}}(u) \cdot F_{\texttt{Min}}$$

The monoid of double matrices

The monoids of double matrices contain the ordered pairs of matrices reachable from the letters.

$$D = \langle (\rho_{\texttt{Max}}(a), \rho_{\texttt{Min}}(a)) \mid a \in \Sigma \rangle \subseteq M_{\texttt{Max}} \times M_{\texttt{Min}}$$

For $n \in \mathbb{N}$, define:

$$D_n = \{(\rho_{\texttt{Max}}(u), \rho_{\texttt{Min}}(u)) \mid |u| \leqslant n\}$$

Given $(A, B) \in D$ and $x \in \mathbb{R}$, define:

$$(A, B) + x = ((A_{p,p'} + x)_{p,p' \in Q_{\texttt{Max}}}, (B_{q,q'} + x)_{q,q' \in Q_{\texttt{Min}}})$$

Given $(A, B), (A', B') \in M_{\texttt{Max}} \times M_{\texttt{Min}}$, define:

$$(A, B) \succcurlyeq (A', B') \quad \text{if} \quad A \leqslant A' \quad \text{and} \quad B' \leqslant B$$

« coarser than » relation

Key lemma: There exists $n \in \mathbb{N}$, such that for all $(A, B) \in D$, there exists $(A', B') \in D_n$ and $x \in \mathbb{R}$ such that

$$(A, B) \succcurlyeq (A', B') + x$$

The construction

Key lemma: There exists $n \in \mathbb{N}$, such that for all $(A, B) \in D$, there exists $(A', B') \in D_n$ and $x \in \mathbb{R}$ such that

$$(A, B) \succcurlyeq (A', B') + x$$

States D_n

Initial state $(\mathtt{Id}_{I_{\mathtt{Max}}}, \mathtt{Id}_{I_{\mathtt{Min}}})$

Restricted using lookahead to states that are part of an accepting run.

Transition upon reading letter a :

$$(A, B) \in D_n \xrightarrow{a \;:\; x} (A', B') \in D_n$$

for some (A', B') and x such that $(A, B) \cdot \rho_D(a) \succcurlyeq (A', B') + x$.

Final map: the one of $\mathtt{Max}$ (on the first component).

Theorem: this unambiguous automaton separates $\mathtt{Max}$ from $\mathtt{Min}$.

Conclusion

Conclusion

Tropical automata can be separated can be separated by unambiguous ones!

In generalizes the result of Lombardy and Mairesse.

What is it good for? Don't know!

Thx