Contents

Open Maps as a Bridge between Algebraic Observational Equivalence and Bisimilarity*

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Abstract

There are two widely accepted notions of *behavioural equivalence*, formalizing the idea of *observational indistinguishability*: *observational equivalence* for algebras (which are models for sequential systems) and *bisimulation equivalence (bisimilarity)* for concurrent processes. In this paper we show that the observational equivalences for standard, partial and regular algebras are bisimulation equivalences. This is done in the setting of *open maps*, proposed in [JNW93] as an abstract approach to behavioural equivalences of processes. The main advantage of the results is capturing the models for sequential and concurrent systems in a uniform framework. In such an abstract setting we formulate the property of *determinism*, shared by all the algebras considered in this paper, and identify some interesting facts about bisimilarity in the deterministic case. All the results for standard, regular and partial algebras are obtained by the applications of a general machinery developed in the paper, which we expect to be applicable also in other contexts.

1 Introduction

Motivations and related research

The concept of *behavioural equivalence* is a fundamental notion in programming methodology, as it seems to capture appropriately the "black box" character of data abstraction. It plays also a central role in the process of stepwise refinement, where each step embodies some design decisions, under the requirement that behaviour must be preserved. This applies to two different notions: observational equivalence of algebras, being models for sequential programs, as well as bisimulation equivalence (bisimilarity) of processes, modeling concurrent or nondeterministic systems[1].

Roughly speaking, two algebras A and B over a signature Σ are *observationally equivalent* with respect to a distinguished set OBS of *observable sorts* if all computations (using operations from Σ) yielding a result in an observable sort produce the same output in both A and B. It means that A and B are indistinguishable by means of computations with observable results. This idea goes back to

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[1]Throughout this paper by *observational equivalence* we mean a behavioural equivalence of algebras whereas term *behavioural equivalence* will be used more generally, for observational equivalence as well as for bisimilarity.

[Mor68], [Hoa72], [GGM76], and was widely studied mainly in the case of standard algebras [Rei81], [ST87], but also in some other frameworks, like regular algebras [BT95] and partial algebras [BT96].

Another approach to behavioural equivalence is the notion of *bisimilarity* of concurrent processes [Mil89], [Par81]. Two processes are bisimilar if there exists a relation of *bisimulation* between them, meaning that every action of one process can be simulated in the other one and vice versa. Bisimilarity is thus a formalization of indistinguishability of processes by observing their actions. Recently a categorical generalization of bisimilarity was proposed, in the setting of *open maps (open morphisms)* [JNW93], enabling a uniform definition of bisimulation across a range of different models for parallel computations. This setting turned out appropriate for defining, among others, strong and weak bisimilarity [Mil89], trace equivalence [Hoa85], testing equivalence [Hen88] and a strong history preserving bisimulation of event structures [VGG89] (see [CN95], [JNW93] for an overview). Open maps can be understood as witnesses of bisimulation between objects, and are defined abstractly to be those morphisms which satisfy a *computation-lifting property*, where a computation of a process is modeled by a morphisms from any of distinguished *observation objects*. Thus bisimilarity is determined (induced) by the choice of a subcategory $\mathcal{P}$ of observation objects. This concept is related to open morphisms in a topos of set-valued functors from $\mathcal{P}^{op}$ [JM94].

Contribution of the paper

Both observational equivalence and bisimilarity play a crucial role in formal models of sequential and concurrent systems, respectively. The undeniable analogies between these approaches are well recognised and much work was done in linking the two notions. This paper is a continuation of this stream of research, but the first one to consider abstract data types and processes commonly in the context of open maps. As some examples of related work we should mention [Gor95], where a characterisation of contextual equivalence of PCF terms was given by means of bisimulation, and [AO93], where authors proposed an approach to semantics of lazy λ-calculus, inspired by bisimulation. Moreover, in [Jac95], [Mal96] some links were observed between bisimulation and observational equivalence for hidden-sorted algebras, viewed as coalgebras. We do not consider here any other abstract depictions of bisimulation (like spans of coalgebra maps), leaving this for further research.

The most important corollary from results stated in this paper is that observational equivalences for some classes of algebras (standard, partial and regular algebras are considered) can be captured in the setting of open maps, and thus can be thought of as abstract bisimilarities.

In Section 3.1 we show that the observational equivalence of standard and regular algebras w.r.t. a set OBS of observable sorts is a bisimilarity induced by the class of observation objects consisting of free objects w.r.t. a forgetful functor $|\_|_{OBS}$. To prove this we formulate an observation (Lemma 3.1 in Section 3 and Theorem 4.3 in Section 4) that bisimilarity defined by means of open maps can be transported via *multiadjunction* in both directions. The notion of *multiadjunction*, introduced in [Die79], is a generalization of adjunction where instead of free object there exists a whole family of "jointly" free objects. While for categories of standard and regular algebras usual adjunctions are sufficient, this does not extend to partial algebras, where the class of free objects is too small to induce the observational equivalence. In Section 4.2 the observational equivalence of partial algebras is proved to be a bisimilarity induced by union of all families of free objects of some multiadjunction. This is done in the category of partial algebras with *strong* partial homomorphisms, similarly as in [BT96]. Such a result is not quite satisfactory, as the choice of strong homomorphisms can not be easily justified. To do the same for partial algebras with all (weak) homomorphisms, we introduce the notion of *pre-adjunction*, being a further relaxation of requirements of multiadjunction. It turns out that every pre-adjunction gives rise to some multiadjunction, when certain necessary conditions are satisfied. We prove that bisimilarity induced by pre-free objects of a pre-adjunction and bisimilarity induced by free objects of the related multiadjunction coincide (Theorem 5.7 in Section 5). This allows us to characterize observational equivalence of partial algebras by means of open maps in the category with weak homomorphisms (in Section 5.1).

The results concerning open maps in multiadjunction and pre-adjunction are formulated for an arbitrary category, and we expect they are applicable in other frameworks than considered in this paper.

The surprising observation is that all bisimilarities studied in this paper (i.e. those induced by adjunction, multiadjunction or pre-adjunction) have a common property that each computation in a target of an open map lifts uniquely to a computation in its source". We call such a bisimilarity *deterministic*. This can be understood as that open morphisms do not reduce nondeterminism of computations. We introduce the notion of deterministic bisimilarity in Section 4.1. Deterministic open maps correspond to the class of *etale maps* ([JM94]) in a topos. An interesting conclusion is that while open maps appeared useful for behavioural equivalence of concurrent or nondeterministic systems, the same role in models for sequential, deterministic systems is played by etale maps.

We are also interested in how large the class of observation objects should be to induce the observational equivalence. For example, in the case of standard and regular algebras it turns out that free algebras over (at most) two-element sets are sufficient, i.e. observations by means of terms over two variables are powerful enough to distinguish non-equivalent algebras. This gives a useful tool in showing equivalence of algebras, as we only have to examine a small category of observation objects. The detailed study of observation subcategories for standard, regular and partial algebras are summarized in Corollaries 3.7, 3.8, 4.13 and 5.18. Moreover, in the case of deterministic bisimulation, the class of observation objects can be reduced by Lemma 4.11 from Section 4.1 stating that closure of the subcategory of observation objects under colimits has no influence on induced class of open maps.

2 Preliminaries

Basic knowledge of category theory is assumed: for the basic notions used here, like functor, colimit and adjunction, we refer to [Mac71]. The definitions of categories of standard, partial and regular algebras can be found in Section 2.1. Observational equivalence is defined in Section 2.2. In Section 2.3 we recall the notion of open maps.

2.1 Standard, partial and regular algebras

In this paper we investigate observational equivalences for standard, partial and regular algebras. Standard algebras, consisting of a set of states (values) and (total) operations on them, are widely accepted models for abstract data types [BL70], [GTW78]. In some applications such models are no sufficient; for example, when the totality of operations is too restrictive, partial algebras are used [BW82], in which operations are partial functions. In many applications a kind of "infinitary" data is needed, for instance to specify a programming language or to ensure that recursive definitions are meaningful – to this aim continuous algebras were studied [TW86], where carriers are cpo's and operations are continuous maps. Category of continuous algebras lacks some nice algebraic properties (for example quotients do not compose), hence the setting of *regular algebras* was proposed [Tiu78], in which not all increasing chains must have least upper bounds, only those definable by finite terms. In [BT95] regular algebras are studied in the context of algebraic specifications.

Despite such a variety of algebras, the observational equivalence is defined uniformly for all of them by means of the greatest congruences being identity on OBS (observable sorts). Two algebras are observationally equivalent if their quotients by such congruences are isomorphic.

We present here basic definitions concerning standard, partial and regular algebras. Many notions, like those of (standard, partial, regular) subalgebra, congruence, quotient, etc., are instances of general definitions for arbitrary concrete category, presented in a uniform setting in [BT96]. The definitions concerning standard many sorted algebras are widely accepted [BL70], [GTW78]. Definitions of partial algebras are based on [BW82] and [BT96], while the presentation of regular algebras follows [BT95].

Standard algebras A signature is a pair $\langle S, F\rangle$, where $S = sorts(\Sigma)$ is a set of sorts and F a set of operation names. Let Σ be a fixed signature throughout this paper, unless stated otherwise. Σ-algebra consists of an S-sorted set, denoted by $|A|$ (*the carrier* of A) and for each $f : s_1 \times \ldots \times s_n \to s$ in Σ, of a function $f_A : |A|_{s_1} \times \ldots \times |A|_{s_n} \to |A|_s$. A Σ-homomorphism $h : A \to B$ is a Σ-sorted function $|h| : |A| \to |B|$ which preserves operations, i.e. for each $f : s_1 \times \ldots \times s_n \to s$ in Σ, $|h|_s(f_A(a_1, \ldots , a_n)) = f_B(|h|_{s_1}(a_1), \ldots , |h|_{s_n}(a_n))$. Let $Alg(\Sigma)$ denote the category of standard Σ-algebras with Σ-homomorphisms. The mapping $|\_|$ is a forgetful functor $|\_| : Alg(\Sigma) \to Set^S$. One can define a family of forgetful functors $|\_|_R : Alg(\Sigma) \to Set^R$, for every set of sorts $R \subseteq S$.

A congruence on A is a many sorted equivalence relation $\sim\ \subseteq |A| \times |A|$ compatible with operations, that is, such that for each $f : s_1 \times \ldots \times s_n \to s$ in Σ and $a_1, a'_1 \in |A|_{s_1}, \ldots , a_n, a'_n \in |A|_{s_n}$, if $a_1 \sim_{s_1} a'_1, \ldots , a_n \sim_{s_n} a'_n$ then $f_A(a_1, \ldots , a_n) \sim_s f_A(a'_1, \ldots , a'_n)$. Given a congruence relation $\sim$, the quotient algebra $A/_\sim$ is defined by $|A/_\sim| = |A|/_\sim$ and $f_{A/_\sim}([a_1]_\sim, \ldots , [a_n]_\sim) = [f_A(a_1, \ldots , a_n)]_\sim$.

A subalgebra of A is any algebra B the carrier of which is a subset of $|A|$ closed under operations and operations of which are equal to operations of A restricted to $|B|$.

The S-sorted set $|T_\Sigma(X)|$ of Σ-terms with variables from X is defined inductively as usual, for any X from Set^S. The relation of satisfaction of an equation $t = t'$, where $t, t' \in T_\Sigma(X)$, in Σ-algebra A under a valuation $v : X \to |A|$, denoted by $A[v] \vDash t = t'$, is defined by $A[v] \vDash t = t'$ iff $t_{A[v]} = t'_{A[v]}$, where $t_{A[v]}$ denotes the value of a term t in algebra A under a valuation $v : X \to |A|$.

The set $|T_\Sigma(X)|$, where X is in Set^S, can be equipped in the canonical way with operations of Σ to form a Σ algebra $T_\Sigma(X)$. Namely, for each $f : s_1 \times \ldots \times s_n \to s$ in Σ, we define $f_{T_\Sigma(X)}(t_1, \ldots , t_n) = f(t_1, \ldots , t_n)$. An algebra $T_\Sigma(X)$ is a free object over X w.r.t. the forgetful functor $|\_|$, hence every function (valuation) $v : X \to |A|$ has a unique extension $v^\# : T_\Sigma(X) \to A$ in $Alg(\Sigma)$; moreover $T_\Sigma(\_)$ can be extended to a functor being the left adjoint to $|\_|$.

Notation: We will overload the notation $T_\Sigma(X)$ using it also for X from Set^R, where $R \subseteq S$; thus $T_\Sigma(\_)$ will denote the left adjoint to any forgetful functor $|\_|_R : Alg(\Sigma) \to Set^R$, returning the carriers of sorts from R. Generally, we will identify category Set^R with the full subcategory of Set^S containing all objects satisfying $X_r = \emptyset$ for all $r \notin R$, ignoring the subtle difference. Moreover, we will use the same notation $|\_|_R$ also for the forgetful functors from categories of partial and regular algebras.

Partial algebras Partial Σ-algebras are like standard Σ-algebras except that their operations can be partial functions. A (weak) partial Σ-homomorphism $h : A \to B$ is a S-sorted function $|h| : |A| \to |B|$ preserving definedness of operations, i.e. for each $f : s_1 \times \ldots \times s_n \to s$ in Σ and $a_1 \in |A|_{s_1}, \ldots , a_n \in |A|_{s_n}$, if $f_A(a_1, \ldots , a_n)$ is defined then $f_B(|h|_{s_1}(a_1), \ldots , |h|_{s_n}(a_n))$ is defined and $|h|_s(f_A(a_1, \ldots , a_n)) = f_B(|h|_{s_1}(a_1), \ldots , |h|_{s_n}(a_n))$. Let $PAlg_w(\Sigma)$ denote the category of partial Σ-algebras with weak partial Σ-homomorphisms.

A partial Σ-homomorphism is *strong* if it not only preserves but also reflects definedness of operations, i.e. $f_A(a_1, \ldots, a_n)$ is defined iff $f_B(|h|_{s_1}(a_1), \ldots, |h|_{s_n}(a_n))$ is defined, for each $f : s_1 \times \ldots \times s_n \to s$ in Σ and $a_1 \in |A|_{s_1}, \ldots, a_n \in |A|_{s_n}$. The subcategory of partial algebras with strong partial Σ-homomorphisms we denote by $PAlg(\Sigma)$.

A partial subalgebra of A is any partial algebra B the carrier of which is a subset of A closed under operations and operations of which are equal to operations of A restricted to $|B|$.

A congruence on A is a many sorted equivalence relation $\sim \subseteq |A| \times |A|$, such that for each operation $f : s_1 \times \ldots \times s_n \to s$ in Σ and $a_1, a'_1 \in |A|_{s_1}, \ldots, a_n, a'_n \in |A|_{s_n}$, whenever $a_1 \sim_{s_1} a'_1, \ldots, a_n \sim_{s_n} a'_n$ then $f_A(a_1, \ldots, a_n)$ and $f_A(a'_1, \ldots, a'_n)$ are both undefined or both defined and $f_A(a_1, \ldots, a_n) \sim_s f_A(a'_1, \ldots, a'_n)$. Given a congruence relation $\sim$, the quotient algebra $A/_\sim$ is defined by $|A/_\sim| = |A|/_\sim$ and $f_{A/_\sim}([a_1]_\sim, \ldots, [a_n]_\sim)$ is defined iff $f_A(a_1, \ldots, a_n)$ is defined, and then $f_{A/_\sim}([a_1]_\sim, \ldots, [a_n]_\sim) = [f_A(a_1, \ldots, a_n)]_\sim$.

The quotient projection and the inclusion of subobjects are strong partial homomorphisms, hence definitions above correspond to general definitions from [BT96] (a concrete category) applied to $PAlg(\Sigma)$.

In the case of partial algebras, the value $t_{A[v]}$ of a term under a valuation $v : X \to |A|$ can be undefined, as operations are allowed to be partial functions. The relation of satisfaction of an equation $t = t'$, for $t, t' \in T_\Sigma(X)$, in a Σ-algebra A under a valuation $v : X \to |A|$ is defined as follows: $A[v] \models t = t'$ iff either both $t_{A[v]}$ and $t'_{A[v]}$ are undefined or both are defined and $t_{A[v]} = t'_{A[v]}$.

Regular algebras An *ordered* Σ*-algebra* is a standard Σ-algebra equipped with a partial order $\leq_s^A$, for each sort $s \in S$ and a distinguished element $\perp_s^A \in |A|$, for each $s \in S$.

The set of regular Σ-terms $|T_\Sigma^\mu(X)|$ with variables from X is defined inductively as usual, with the only additional requirement:

$$\text{for any } t \in |T_\Sigma^\mu(X)|_s \text{ and a variable } z \in X_s \text{ of the same sort } s,\ \mu z.t \in |T_\Sigma^\mu(X)|.$$

Similarly, the inductive definition of the value $t_{A[v]}$ of a term under a valuation $v : X \to |A|$ needs one more case. For any $t \in |T_\Sigma^\mu(X \cup \{z : s\})|_s$, where z is a distinguished variable of sort s, put:

- $t^0_{A[v]}(\perp) = \perp_s^A$,
- for $i \geq 0$, $t^{i+1}_{A[v]}(\perp) = t_{A[v_i]}$, where $v_i : X \cup \{z : s\} \to |A|$ extends v by $v_i(z) = t^i_{A[v]}(\perp)$.

Now, $(\mu z.t)_{A[v]}$ is defined if $t^i_{A[v]}(\perp)$ are defined, for all $i \geq 0$, $t^i_{A[v]}(\perp) \leq t^{i+1}_{A[v]}(\perp)$, for all $i \geq 0$, and the least upper bound $\bigsqcup_{i \geq 0} t^i_{A[v]}(\perp)$ exists in $|A|_s$. Then $(\mu z.t)_{A[v]} = \bigsqcup_{i \geq 0} t^i_{A[v]}(\perp)$.

An ordered algebra A is a regular Σ-algebra if it satisfies following two conditions:

- (completeness): for all terms $t \in |T_\Sigma^\mu(X)|$ and valuations $v : X \to |A|$, the value $t_{A[v]}$ of t under v in A is defined,
- (continuity): for all terms $t \in |T_\Sigma^\mu(X \cup \{y : s\})|_{s'}$, $q \in |T_\Sigma^\mu(X \cup \{z : s\})|_s$ and valuations $v : X \to |A|$, $t_{A[v_i]} \leq t_{A[v_{i+1}]}$, for $i \geq 0$, and $t_{A[v']} = \bigsqcup_{i \geq 0} t_{A[v_i]}$, where $v' : X \cup \{y : s\} \to |A|$ extends v by $v'(y) = (\mu z.q)_{A[v]}$ and for $i \geq 0$, $v_i : X \cup \{y : s\} \to |A|$ extends v by $v_i(z) = q^i_{A[v]}(\perp)$.

By a regular Σ-homomorphism $h : A \to B$ we mean any function $|h| : |A| \to |B|$ such that for all terms $t \in |T_\Sigma^\mu(X)|$ and valuations $v : X \to |A|$, $|h|(t_{A[v]}) = t_{B[v;h]}$. Regular algebras together with regular homomorphisms form a category, which we call $RAlg(\Sigma)$.

A regular subalgebra of A is any regular algebra B the carrier of which is a subset of A and for all terms $t \in |T^{\mu}_{\Sigma}(X)|$ and valuations $v : X \to |B|$, $t_{B[v]} = t_{A[v]}$. It is easy to see that all operations of B are restrictions of operations of A to $|B|$ and moreover $\perp^B_s = \perp^A_s$ for each sort s.

The kernel of a many sorted function $f : X \to Y$ is a set of pairs $\{(x, x') \mid f(x) = f(x')\}$. By a regular congruence on A we mean a kernel of any regular Σ-homomorphism $h : A \to B$. Given a regular congruence $\sim$, the quotient algebra $A/_\sim$ is defined (for each sort s) by $|A/_\sim|_s = |A|_s/_\sim$, $\perp^{A/\sim}_s = \perp^A_s$, $[a]_\sim \leq^{A/\sim}_s [a']_\sim \Leftrightarrow a \sqsubseteq a'$ (where $\sqsubseteq$ is a regular pre-congruence inducing $\sim$, i.e $\sim = \sqsubseteq \cap \sqsubseteq^{-1}$) and $f_{A/\sim}([a_1]_\sim, \ldots, [a_n]_\sim) = [f_A(a_1, \ldots, a_n)]_\sim$. For details we refer to [BT95].

2.2 Observational equivalence

Throughout the paper let us fix a subset $OBS \subseteq S$ of *observable sorts*. By *observational morphisms* we mean the (standard, partial, regular) Σ-homomorphisms which are bijective on observable sorts.

In this section we define the observational equivalence for standard, partial and regular algebras, called here commonly Σ-algebras. For a Σ-algebra A and $X \subseteq |A|$, by $\langle X \rangle_A$ we denote the least (standard, partial, regular) subalgebra of A whose carrier includes X. $\langle X \rangle_A$ will be called *the subalgebra of A generated by X*. By a *partial congruence* on a Σ-algebra A we mean any congruence on a subalgebra of A.

For a Σ-algebra A we define the partial congruence $\sim^{OBS}_A$ as follows. The domain of $\sim^{OBS}_A$ is the carrier $|A_{OBS}|$ of $A_{OBS} = \langle |A|_{OBS} \rangle_A$. Then we define $\sim^{OBS}_A$ as the greatest congruence on A_{OBS} being identity on observable sorts. The partial equivalence $\sim^{OBS}_A$ is a relation of *indistinguishability* of elements of A [BT96]. There exist explicit characterizations of $\sim^{OBS}_A$ as the relations of indistinguishability by means of observable Σ-contexts (see [BHW94] for standard algebras, [BT96] for partial algebras and [BT95] for regular algebras).

Definition 2.1 *An observational equivalence of (standard, partial, regular) algebras is an equivalence relation $\equiv_{OBS}$* factorized *by the family*[2] $\{\sim^{OBS}_A\}_{A \in |Alg(\Sigma)|}$ *of partial congruences, that is to say:*

$$A \equiv_{OBS} B \quad \textit{iff} \quad A/_{\sim^{OBS}_A} \simeq B/_{\sim^{OBS}_B}.$$

The quotient $A/_{\sim^{OBS}_A}$ of A by the partial congruence $\sim^{OBS}_A$ is defined as the quotient of A_{OBS} by $\sim^{OBS}_A$.

Equivalence $\equiv_{OBS}$ in the categories of standard, partial and regular algebras, respectively, can be characterized as follows:

Theorem 2.2 (standard algebras, [BHW94], regular algebras, [BT95]) *For any standard (regular) Σ-algebras A and B, $A \equiv_{OBS} B$ iff for some set $X \subseteq Set^{OBS}$ of variables of observable sorts and surjective valuations $v_A : X \to |A|_{OBS}$ and $v_B : X \to |B|_{OBS}$, for all terms $t, t' \in |T_\Sigma(X)|_o$ ($|T^{\mu}_{\Sigma}(X)|_o$) of observable sort $o \in OBS$, the following holds:*

$$A[v_A] \models t = t' \quad \textit{iff} \quad B[v_B] \models t = t'.$$

Theorem 2.3 (partial algebras, [BT96]) *For any partial Σ-algebras A and B, $A \equiv_{OBS} B$ iff for some set $X \subseteq Set^{OBS}$ of variables of observable sorts and surjective valuations $v_A : X \to |A|_{OBS}$ and $v_B : X \to |B|_{OBS}$, we have:*

[2] Throughout the paper $|\mathcal{C}|$ denotes the class of all objects of a category $\mathcal{C}$, thus we deliberately overload the symbol $|\_|$.

- *for all terms* $t \in |T_\Sigma(X)|$, $t_{A[v_A]}$ *is defined* *iff* $t_{B[v_B]}$ *is defined*
- *for all terms* $t, t' \in |T_\Sigma(X)|_o$ *of observable sort* $o \in OBS$,

$$A[v_A] \models t = t' \quad \textit{iff} \quad B[v_B] \models t = t'.$$

2.3 Bisimilarity from open maps

In this section we recall from [JNW93] the general notion of bisimulation (for any category of models), defined by means of *open morphisms*. This is a generalization of the strong bisimulation for transition systems [Mil89].

Let $\mathcal{N}$ be a category of models of computation. In $\mathcal{N}$ we choose a subcategory (not necessarily full) of *observation objects*, which we denote by $\mathcal{Q}$. A morphism $p : O \to A$ from an observation object $O \in |\mathcal{Q}|$ can be seen as *a computation* in A. A morphism $h : A \to B$ in $\mathcal{N}$ should be intuitively thought of as a simulation of A in B in the following sense: h transforms every computation $p : O \to A$ in A into a computation $p; h : O \to B$ in B. Moreover any morphism $m : O_1 \to O_2$ in $\mathcal{Q}$ making $p = m; q$, for two computations p, q, means intuitively that a "larger" computation q is an extension of p (via m).

Definition 2.4 (Open morphisms) *A morphism* $h : A \to B$ *in* $\mathcal{N}$ *is* $\mathcal{Q}$-open *if for any morphism* $m : O_1 \to O_2$ *in* $\mathcal{Q}$, $p : O_1 \to A$, $q : O_2 \to B$ *in* $\mathcal{N}$, *whenever the diagram*

$$\begin{tikzcd} O_1 \arrow[r, "p"] \arrow[d, "m"'] & A \arrow[d, "h"] \\ O_2 \arrow[r, "q"'] & B \end{tikzcd} \tag{1}$$

commutes, i.e. $p; h = m; q$, *there exists a morphism* $r : O_2 \to A$ *in* $\mathcal{N}$ *such that the two triangles in the diagram (1) commute, i.e.* $p = m; r$ *and* $q = r; h$.

Such a morphism r will be called a *diagonal* morphism.

Definition 2.5 (Bisimilarity) *Two models* A *and* B *in* M *are said to be* $\mathcal{Q}$-bisimilar, *denoted by* $A \sim_{\mathcal{Q}} B$, *if there exists in* $\mathcal{N}$ *a span of open morphisms* $A \longleftarrow Z \longrightarrow B$ *from a common object* $Z \in |\mathcal{N}|$.

We often omit the prefix $\mathcal{Q}$-, when it is obvious from a context. The $\mathcal{Q}$-open morphisms form a subcategory of $\mathcal{N}$, as they include all the identity morphisms and are closed under composition. It can be easily checked that all isomorphisms are $\mathcal{Q}$-open. $\mathcal{Q}$-bisimilarity is not necessarily an equivalence relation (transitivity !). The sufficient condition for bisimilarity to be an equivalence is for example the existence of all pullbacks in $\mathcal{N}$ as pullbacks of open morphisms are open – the proof can be found in [JNW93]. The notion of bisimilarity is a generalization of the strong bisimilarity between transition systems. In [JNW93] it was proved that strong bisimilarity coincides with $Bran_L$-bisimilarity in the category of transition systems, where the subcategory $Bran_L$ consists of finite "linear" transition systems.

Throughout the paper we implicitly rely on the observation that the larger an observation subcategory $\mathcal{Q}$ is, the fewer morphisms are $\mathcal{Q}$-open: if $\mathcal{P}$ is a subcategory of $\mathcal{Q}$ then every $\mathcal{Q}$-open morphism is $\mathcal{P}$-open.

3 Bisimilarity in adjunction

In this section we state some general result leading to the conclusion that observational equivalence of standard and regular algebras coincide with the bisimilarity defined by means of open morphisms w.r.t. a natural class of observation objects.

Let us assume that we are given a functor $G : \mathcal{M} \to \mathcal{O}$ from the category $\mathcal{M}$ of models to the category $\mathcal{O}$ having the left adjoint $F : \mathcal{O} \to \mathcal{M}$. The category $\mathcal{O}$ can be intuitively understood here as the category of observation objects, which this time are outside $\mathcal{M}$. Throughout this section let us fix the subcategory $\mathcal{P}$ of $\mathcal{O}$. As the subcategory of observation objects in $\mathcal{M}$ we choose the image $F(\mathcal{P})$ of the functor F. We will study especially the case when $\mathcal{P} = \mathcal{O}$.

The following facts are special cases of Theorem 4.3 and Lemma 4.4, respectively:

Lemma 3.1 *A morphism $h : A \to B$ in $\mathcal{M}$ is $F(\mathcal{P})$-open iff $G(h)$ is $\mathcal{P}$-open in $\mathcal{O}$.*

Corollary 3.2 *A morphism $h : A \to B$ in $\mathcal{M}$ is $F(\mathcal{O})$-open iff $G(h)$ is an isomorphism in $\mathcal{O}$.*

Lemma 3.1 will be a basic tool in analyzing bisimulations for standard and regular algebras. Essentially it states that open bisimulation is transported via adjunction in both directions. In Corollary 3.2 we deal with the case when $\mathcal{P}$ is the whole category $\mathcal{O}$: the $F(\mathcal{O})$-open morphisms in $\mathcal{M}$ are precisely those transported by G to isomorphisms.

Some weaker version of Lemma 3.1, concerning only coreflections (i.e. adjunctions with unit being a natural isomorphism), was already proved in [JNW93].

For any subcategory $\mathcal{S}$ of $\mathcal{M}$ let $[\mathcal{S}]$ denote the smallest full subcategory of $\mathcal{M}$ containing all the objects of $\mathcal{S}$ (in fact $\mathcal{S}$ and $[\mathcal{S}]$ can differ only on morphisms). The choice of observation objects as an image subcategory $F(\mathcal{O})$ of $\mathcal{M}$ is not the only natural - we could also take (usually different) full subcategory $[F(\mathcal{O})]$, having the same objects but in general more morphisms. It turns out that both choices give the same result, which we prove in Lemma 4.5 in Section 4. The following is a specialization of Lemma 4.5 for the adjunction:

Corollary 3.3 *A morphism $h : A \to B$ in $\mathcal{M}$ is $F(\mathcal{O})$-open iff it is $[F(\mathcal{O})]$-open.*

The straightforward conclusion from Corollary 3.2:

Corollary 3.4 *Let A, B be objects in $\mathcal{M}$. If A and B are $F(\mathcal{O})$-bisimilar then $G(A)$ and $G(B)$ are isomorphic.*

Proof: There exists in $\mathcal{M}$ a span $A \xleftarrow{f} C \xrightarrow{g} B$, where f and g are open, thus $G(f)$, $G(g)$ are isomorphisms. Then $(G(f)^{-1}; G(g)) : G(A) \to G(B)$ is also an isomorphism. □

The implication in an opposite direction does not hold, in general. Let us consider a very special situation, studied in the context of open maps in [JNW93], when the adjunction between F and G is a coreflection. We have now a characterization of $F(\mathcal{O})$-bisimilarity:

Proposition 3.5 *If the adjunction between F and G is a coreflection then for any objects A, B in $\mathcal{M}$, A and B are $F(\mathcal{O})$-bisimilar iff $G(A)$ and $G(B)$ are isomorphic.*

Proof: The "only if" part follows easily from Corollary 3.4, so we only need to prove the "if" implication. Assume that $G(A)$ and $G(B)$ are isomorphic. First we show that for any object X of $\mathcal{M}$, the counit $\varepsilon_X = id^{\#}_{G(X)}$ is $F(\mathcal{O})$-open, i.e. (by Corollary 3.2) that $G(\varepsilon_X)$ is iso. Let $\eta : Id_{\mathcal{O}} \to G \circ F$

denote the unit natural transformation. We have $\eta_{G(X)}; G(\varepsilon_X) = id_{G(X)}$, hence $G(\varepsilon_X) = {\eta_{G(X)}}^{-1}$ as $\eta_{G(X)}$ is iso. Thus, $G(\varepsilon_X)$ is an isomorphism as well.

Now A and B are bisimilar because we have a span of open morphisms:

$$A \xleftarrow{\varepsilon_A} F(G(A)) \simeq F(G(B)) \xrightarrow{\varepsilon_B} B$$

as all isomorphisms are open. □

3.1 Standard and regular algebras

We separated the special case of adjunction from the more general situation (with which we deal in Section 4) since there are some classical and important examples of such a situation, namely standard and regular algebras. Recall that by Σ we denote an arbitrary fixed signature and by OBS we denote a subset of observable sorts.

Standard algebras As an example consider the category $Alg(\Sigma)$ of standard algebras together with the forgetful functor $|\_|_{OBS} : Alg(\Sigma) \to Set^{OBS}$ returning the carrier sets of sorts in OBS. The functor $|\_|_{OBS}$ has the left adjoint T_Σ. We conclude from Corollary 3.2 that the $T_\Sigma(Set^{OBS})$-open morphisms of standard algebras are precisely the observational morphisms. This way we obtain the characterization of observational equivalence by means of $T_\Sigma(Set^{OBS})$-open maps:

Corollary 3.6 *The observational equivalence of standard algebras coincides with $T_\Sigma(Set^{OBS})$-bisimilarity.*

Proof: We base the proof on Theorem 4.13 from [BT96] (or Theorem 28 in the short version of [BT96]), which states that standard (regular, partial) algebras are observationally equivalent iff there exists an *observational correspondence* between them. Moreover, all spans in $Alg(\Sigma)$ are correspondences, hence two algebras A and B are observationally equivalent iff there exists an observational span $A \xleftarrow{f} C \xrightarrow{g} B$ to them, i.e. a span in which f and g are observational. But this is precisely another formulation of A and B being $T_\Sigma(Set^{OBS})$-bisimilar, due to the fact that the $T_\Sigma(Set^{OBS})$-open morphisms of standard algebras are precisely the observational morphisms. □

It is interesting to observe that a subcategory of observation objects can be reduced while still inducing the same bisimilarity relation. Any subcategory $T_\Sigma(\mathcal{P})$ of $T_\Sigma(Set^{OBS})$ is a candidate when $\mathcal{P}$ is chosen large enough to "distinguish" between non-isomorphic objects in Set^{OBS}, i.e. to make them non-$\mathcal{P}$-bisimilar. Surprisingly, very small subcategory $\mathbf{2}_{OBS}$ is sufficient, where $\mathbf{2}_{OBS}$ consists of only those objects X in Set^{OBS} for which X_s, for any $s \in OBS$, is empty or of cardinality two. We obtain the following fact as a corollary from Lemma 3.1 and Corrolaries 3.2 and 3.3:

Corollary 3.7 *The observational equivalence of standard algebras coincides with $\mathcal{Q}$-bisimilarity, for any subcategory $\mathcal{Q}$ of $[T_\Sigma(Set^{OBS})]$ containing $T_\Sigma(\mathbf{2}_{OBS})$.*

Proof: We only have to show (by Lemma 3.1) that $\mathbf{2}_{OBS}$-bisimilarity in Set^{OBS} coincides with Set^{OBS}-bisimilarity, i.e. that two OBS-sorted sets X, Y are $\mathbf{2}_{OBS}$-bisimilar iff they are isomorphic. This coincidence (only one implication is non-trivial) can be easily verified, as in the proof of Lemma 5.17. □

Corollary 3.7 can be interpreted as follows: terms of observable sorts with only two variables are powerful enough to distinguish algebras, that are not observationally equivalent. Observational equivalence of standard algebras is thus very "robust" to changes of subcategory of observation objects.

Note When there are no Σ-algebras with empty carrier of sort $s \in OBS$ in our category, we can reduce $\mathbf{2}_{OBS}$ by omitting sets empty on s. Thus, when signature Σ excludes empty carrier set on any observable sort, $\mathbf{2}_{OBS}$ can contain precisely one object (up to isomorphism).

Regular algebras The forgetful functor $|\_|_{OBS} : RAlg(\Sigma) \to Set^{OBS}$ has the left adjoint $T^{\mu}_{\Sigma}(\_)$; free regular algebras are algebras of so called regular Σ-trees [Tiu78]. We obtain hence a conclusion, similarly as for standard algebras:

Corollary 3.8 *The observational equivalence of regular algebras coincides with $\mathcal{Q}$-bisimilarity, for any subcategory $\mathcal{Q}$ of $[T^{\mu}_{\Sigma}(Set^{OBS})]$ containing $T^{\mu}_{\Sigma}(\mathbf{2}_{OBS})$.*

Proof: The argument is a conclusion from Theorem 4.13 (Theorem 28) of [BT96], similarly as in the case of standard algebras. □

4 Bisimilarity in multiadjunction

In Section 3 we considered bisimilarity when an adjunction is given between categories of models and observation objects. In this section we assume the existence of *left multiadjoint* [Die79] to $G : \mathcal{M} \to \mathcal{O}$, where instead of one free object we have a family of free objects. Let $G : \mathcal{M} \to \mathcal{O}$ be arbitrary fixed functor throughout this section. The definition below is adapted from [Die79]:

Definition 4.1 (Left multiadjoint, multiadjunction) *Let X be any object of category $\mathcal{O}$. Let $F(X) = \{F_i(X)\}_{i \in \mathcal{I}_X}$ and $\eta_X = \{\eta^i_X : X \to G(F_i(X))\}_{i \in \mathcal{I}_X}$ be families of objects of $\mathcal{M}$ and morphisms of $\mathcal{O}$, respectively, both indexed by the same set $\mathcal{I}_X$. We say that a pair $\langle F(X), \eta_X\rangle$ is* a family of free objects over X *if for any object A in $\mathcal{M}$ and any morphism $f : X \to G(A)$ there exists a unique pair $\langle i, f^{\#} : F_i(X) \to A\rangle$ where $i \in \mathcal{I}_X$ and $f^{\#}$ is a morphism in $\mathcal{M}$, such that $\eta^i_X ; G(f^{\#}) = f$.*

If for every object X in $\mathcal{O}$ there exists a family of free objects $\langle F(X), \eta_X\rangle$ over X, we say that the pair of families $\langle F = \{F(X)\}_{X \in |\mathcal{O}|}, \eta = \{\eta_X\}_{X \in |\mathcal{O}|}\rangle$ is the left multiadjoint *to G and that $\langle F, \eta\rangle$ and G form a* multiadjunction.

Assume in this section that G has left multiadjoint $\langle F, \eta\rangle$. In [Die79] it was proved that the left multiadjoint is determined uniquely up to isomorphism. The main difference between multiadjunction and adjunction is that we have now a bijection

$$(\_)^{\#_{X,A}} : \mathcal{O}(X, G(A)) \to \bigcup_{i \in \mathcal{I}_X} \mathcal{M}(F_i(X), A), \tag{2}$$

for any X in $\mathcal{O}$ and A in $\mathcal{M}$, instead of a bijection $\mathcal{O}(X, G(A)) \to \mathcal{M}(F(X), A)$. As usual we suppress the index of $\#$, hoping that this should not not lead to an ambiguity. It is easy to observe that when all index sets $\mathcal{I}_X$ are singletons we obtain a usual adjunction.

There is no unique way to define "$F(f)$" for a morphism $f : X \to Y$ in $\mathcal{O}$. Instead we can naturally define a family of morphisms $\{F_j(f)\}_{j \in \mathcal{I}_Y}$, where for $j \in \mathcal{I}_Y$,

$$F_j(f) := (f ; \eta^j_Y)^{\#} : F_i(X) \to F_j(Y) \tag{3}$$

(for certain $i \in \mathcal{I}_X$). This way we obtain a function $\mathcal{I}_f : \mathcal{I}_Y \to \mathcal{I}_X$ (actually we obtain a functor $\mathcal{I}\_ : \mathcal{O}^{op} \to Set$) such that $\mathcal{I}_f(j) = i$ iff the domain of $F_j(f)$ is $F_i(X)$. Moreover the following "naturality" property of η holds:

$$\eta_X^{\mathcal{I}_f(j)}; G(F_j(f)) = f; \eta_Y^j$$

for any morphism $f : X \to Y$ in $\mathcal{O}$ and $j \in \mathcal{I}_Y$. The following lemma will be useful in the sequel. It indicates that the notion of multiadjunction is quite close to adjunction:

Lemma 4.2 *$f^\#; g = (f; G(g))^\#$, for any $f : X \to G(Y)$ in $\mathcal{O}$ and $g : Y \to Z$ in $\mathcal{M}$.*

Proof: Let $f^\# : F_i(X) \to Y$, i.e. $F_i(X)$ is a domain of $f^\#$. It is enough to show that $\eta_X^i; G(f^\#; g) = f; G(g)$. We have

$$\eta_X^i; G(f^\#; g) = \eta_X^i; G(f^\#); G(g) = f; G(g),$$

which yields the result. □

It follows from Lemma 4.2 that morphisms $f^\#$ and $(f; G(g))^\#$ have the same domain object.

Let $\mathcal{P}$ be a fixed subcategory of $\mathcal{O}$ in this section. As a subcategory of observation objects in $\mathcal{M}$ we take now the subcategory $F(\mathcal{P})$ defined as follows: objects are $\{F_i(X)\}_{X \in |\mathcal{P}|, i \in \mathcal{I}_X}$, morphisms are $\{F_j(f)\}_{f : X \to Y \in |\mathcal{P}|, j \in \mathcal{I}_Y}$. This obviously defines a subcategory, as the morphisms are closed under composition and contain all identities. Theorem 4.3 states that openness is transported via multiadjunction in both directions.

Theorem 4.3 *A morphism $h : A \to B$ in $\mathcal{M}$ is $F(\mathcal{P})$-open iff $G(h)$ is $\mathcal{P}$-open in $\mathcal{O}$.*

Proof: $\Leftarrow$: Let $h : A \to B$ be any morphism in $\mathcal{M}$ and assume $G(h)$ is $\mathcal{P}$-open. Consider a commuting square:

$$\begin{tikzcd} F_i(X) \arrow[r, "p^\#"] \arrow[d, "F_j(m)"'] & A \arrow[d, "h"] \\ F_j(Y) \arrow[r, "q^\#"'] \arrow[ur, dashed, "r^\#"] & B \end{tikzcd} \tag{4}$$

where X, Y are objects of $\mathcal{O}$, $m : X \to Y$ is a morphism of $\mathcal{O}$. We will find a morphism $r : Y \to |A|$ such that $r^\#$ makes both triangles above commute. First observe that the square commutes in the following diagram in $\mathcal{O}$:

$$\begin{tikzcd} X \arrow[r, "p"] \arrow[d, "m"'] & G(A) \arrow[d, "G(h)"] \\ Y \arrow[r, "q"'] \arrow[ur, dashed, "r"] & G(B) \end{tikzcd} \tag{5}$$

Diagram (4) commutes, i.e. $p^{\#}; h = F_j(m); q^{\#}$, thus $\eta^i_X; G(p^{\#}; h) = \eta^i_X; G((m; \eta^j_Y)^{\#}; q^{\#})$, from which we conclude $p; G(h) = m; \eta^j_Y; G(q^{\#})$ and then $p; G(h) = m; q$, i.e. diagram (5) commutes. $G(h)$ is $\mathcal{P}$-open so there exists a diagonal morphism $r : Y \to G(A)$ such that $r; G(h) = q$ and $m; r = p$ in diagram (5). We will show that $r^{\#}$ is a diagonal morphism for diagram (4). First note that the domain of $r^{\#}$ is the same as the domain of $q^{\#}$, as $r^{\#}; h = (r; G(h))^{\#} = q^{\#}$ by Lemma 4.2 (we proved here also that bottom-right triangle in diagram (4) commutes). Now we only need to show $F_j(m); r^{\#} = p^{\#}$. We have $\eta^i_X; G(F_j(m); r^{\#}) = m; \eta^j_Y; G(r^{\#}) = m; r = p$ which gives $F_j(m); r^{\#} = p^{\#}$.

$\Rightarrow$: The idea of the proof is analogous to the "if" part. Let $h : A \to B$ be any $F(\mathcal{P})$-open morphism in $\mathcal{M}$. Consider a commuting square (5); now we are looking for a diagonal morphism $r : Y \to G(A)$. First, we lift this square to the commutative square as in (4). We have $\eta^i_X; G(p^{\#}; h) = p; G(h)$ and $\eta^k_X; G(F_j(m); q^{\#}) = m; q$, where $F_j(m) : F_k(X) \to F_j(Y)$. Since the square in (5) commutes, $p^{\#}; h = (p; G(h))^{\#} = (m; q)^{\#} = F_j(m); q^{\#}$, so we get a commutative square in (4) (and in particular $k = i$). Now, since h is $F(\mathcal{P})$-open, there exists a diagonal morphism $r^{\#} : F_j(Y) \to A$ in diagram (4) being necessarily an image under bijection $(\_)^{\#}$ of some morphism $r : Y \to G(A)$. We will show r makes both triangles in diagram (5) commute. First $r; G(h) = \eta^j_Y; G(r^{\#}); G(h) = \eta^j_Y; G(q^{\#}) = q$. Second $m; r = m; \eta^j_Y; G(r^{\#}) = \eta^i_X; G(F_j(m); r^{\#}) = \eta^i_X; G(p^{\#}) = p$. $\square$

For $\mathcal{P}$ being the whole category $\mathcal{O}$ we obtain the following corollary from Theorem 4.3:

Lemma 4.4 *A morphism $h : A \to B$ in $\mathcal{M}$ is $F(\mathcal{O})$-open iff $G(h)$ is an isomorphism in $\mathcal{O}$.*

Proof: We will only prove: f is an isomorphism in $\mathcal{O}$ iff f is $\mathcal{O}$-open. Then the result follows easily by Theorem 4.3. Assumed f is $\mathcal{O}$-open, we construct f^{-1} as a diagonal morphism in the following diagram:

$$\begin{tikzcd} A \arrow[r, "id_A"] \arrow[d, "f"'] & A \arrow[d, "f"] \\ B \arrow[r, "id_B"'] \arrow[ur, "f^{-1}"] & B \end{tikzcd} \tag{6}$$

By $\mathcal{O}$-openness of f such diagonal morphism exists and $f; f^{-1} = id_A$ and $f^{-1}; f = id_B$. $\square$

Lemma 4.5 *A morphism $h : A \to B$ in $\mathcal{M}$ is $F(\mathcal{O})$-open iff it is $[F(\mathcal{O})]$-open.*

Proof: The "if" part is obvious, as $[F(\mathcal{O})]$-openness is stronger. Instead of proving directly the "only if" implication, we will show (based on Lemma 4.4) that h is $[F(\mathcal{O})]$-open whenever $G(h)$ is iso in $\mathcal{O}$. Consider any commuting square:

$$\begin{tikzcd} F_i(X) \arrow[r, "p^{\#}"] \arrow[d, "m^{\#}"'] & A \arrow[d, "h"] \\ F_j(Y) \arrow[r, "q^{\#}"'] \arrow[ur, dashed, "r^{\#}"] & B \end{tikzcd}$$

where X, Y are objects of $\mathcal{O}$, $m : X \to G(F_j(Y))$ is a morphism of $\mathcal{O}$. We will show that $r^{\#}$ is a diagonal morphism where $r := q; G(h)^{-1}$, namely

i) $r^\#; h = q^\#$,

ii) $m^\#; r^\# = p^\#$,

i) From the definition of r we have $r; G(h) = q$. Now by Lemma 4.2 we have $r^\#; h = (r; G(h))^\# = q^\#$. Note that this also means that domains of $r^\#$ and $q^\#$ are the same.

ii) The square in diagram above commutes, i.e. $m^\#; q^\# = p^\#; h$. Hence by i) we have $m^\#; r^\#; h = p^\#; h$. As G is a functor, then $G(m^\#; r^\#); G(h) = G(p^\#); G(h)$, and because $G(h)$ is iso (hence mono), we conclude: $G(m^\#; r^\#) = G(p^\#)$. Now $\eta_X; G(m^\#; r^\#) = \eta_X; G(p^\#) = p$, which implies $m^\#; r^\# = p^\#$. □

In Lemma 4.10 in Section 4.1, equipped with a notion of determinism, we prove a result similar to Lemma 4.5 for a wide class of subcategories $\mathcal{P}$ of $\mathcal{O}$.

4.1 Determinism

Consider a commuting diagram

$$\begin{tikzcd} F_i(X) \arrow[r, "p^\#"] \arrow[d, "F_j(m)"'] & A \arrow[d, "h"] \\ F_j(Y) \arrow[r, "q^\#"'] \arrow[ur, "r^\#"] & B \end{tikzcd}$$

in $\mathcal{M}$ for any $F(\mathcal{O})$-open morphism h and any morphism $m : X \to Y$ in $\mathcal{O}$. It is interesting to note that the diagonal morphism $r^\#$ making both triangles commute is unique. Moreover, this morphism is determined by only h and $q^\#$, i.e. replacing $p^\#$ and/or $F_j(m)$ by any other morphisms does not influence $r^\#$:

Observation 4.6 *For any $F(\mathcal{O})$-open morphism $h : A \to B$ in $\mathcal{M}$ and $q : Y \to G(B)$ in $\mathcal{O}$ there exists at most one morphism $r : Y \to G(A)$ in $\mathcal{O}$ such that $r^\#; h = q^\#$.*

This means that two different computations in A cannot be simulated by the same computation in B. Intuitively speaking, open morphisms can not reduce nondeterminism of computations.

The diagonal morphisms need not to be uniquely determined in general. For example in the category of transition systems with subcategory $Bran_L$ of observation objects ([JNW93]) there can exist in general more than one diagonal morphism in a commuting square (1).

The observation above leads to the general notion of determinism of open morphisms. In the following let $\mathcal{N}$ denote some arbitrary category $\mathcal{N}$ of models and let $\mathcal{Q}$ denote any subcategory of $\mathcal{N}$.

Definition 4.7 *We say that $\mathcal{Q}$-bisimilarity in $\mathcal{N}$ is* deterministic *if any $\mathcal{Q}$-open morphism h has exactly one diagonal morphism in every commuting diagram (1) in Section 2.3. Moreover $\mathcal{Q}$-bisimilarity will be called* strongly deterministic *if for any $\mathcal{Q}$-open morphism $h : A \to B$ in $\mathcal{N}$ and any computation $q : O \to B$ in $\mathcal{N}$ there exists at most one computation $r : O \to A$ in $\mathcal{N}$ such that $r; h = q$.*

Note The notion of deterministic bisimilarity corresponds closely to the *etale maps* studied for example in [JM94]. In this paper we concentrate on the stronger notion, as bisimilarities in all considered categories of algebras are strongly deterministic.

In Observation 4.6 we noticed that $F(\mathcal{O})$-bisimilarity in $\mathcal{M}$ is strongly deterministic for any left multiadjoint F. In Lemma 4.8 we deal with an arbitrary subcategory $\mathcal{P}$ of observation objects:

Lemma 4.8 *Assume $\mathcal{P}$ is a subcategory of $G(\mathcal{M})$. If $\mathcal{P}$-bisimilarity in $G(\mathcal{M})$ is strongly deterministic then $F(\mathcal{P})$-bisimilarity in $\mathcal{M}$ is so as well.*

Proof: Having the bijection (2) it is enough to prove that

$$r; G(h) = s; G(h) \ \text{ if } \quad r^{\#}; h = s^{\#}; h,$$

for any computations r and s in $\mathcal{O}$ with a domain object from $\mathcal{P}$ and any $F(\mathcal{P})$-open morphism h in $\mathcal{M}$. Now by Lemma 4.4 $r^{\#}; h = (r; G(h))^{\#}$ and similarly $s^{\#}; h = (s; G(h))^{\#}$, from which the implication above follows easily. □

The following definition will be useful in this section:

Definition 4.9 *We say that $\mathcal{N}$ is* initialized *(w.r.t. $\mathcal{Q}$) if for any $\mathcal{Q}$-open morphism $h : A \to B$, any object O_2 from $\mathcal{Q}$ and a computation $q : O_2 \to B$ there exists an object O_1 in $\mathcal{Q}$ together with computation $p : O_1 \to A$ and a morphism $m : O_1 \to O_2$ in $\mathcal{Q}$ such that diagram (1) in Section 2.3 commutes, i.e. $p; h = m; q$.*

For example, when $\mathcal{N}$ has the initial object $\mathcal{I}$ and $\mathcal{I}$ belongs to $\mathcal{Q}$ together with all morphisms from $\mathcal{I}$ to objects of $\mathcal{Q}$, then $\mathcal{N}$ is initialized w.r.t. $\mathcal{Q}$. The unique morphism from $\mathcal{I}$ to A can be thought of as the "empty computation" in A.

In the deterministic case some closures of observation subcategory do not change the induced bisimilarity. First "closure" property, stated in Lemma 4.10, is a strengthening of Lemma 4.5.

Lemma 4.10 *Assume that $\mathcal{N}$ is initialized w.r.t. $\mathcal{Q}$ and $\mathcal{Q}$-bisimilarity in $\mathcal{N}$ is strongly deterministic. Then a morphism in $\mathcal{N}$ is $\mathcal{Q}$-open iff it is $[\mathcal{Q}]$-open.*

Proof: Let h be any $\mathcal{Q}$-open morphism in $\mathcal{N}$. We will show h to be $[\mathcal{Q}]$-open, i.e. we will construct a diagonal morphism r in any commuting square:

$$\begin{tikzcd} X \arrow[r, "p"] \arrow[d, "m"'] & A \arrow[d, "h"] \\ Y \arrow[r, "q"'] & B \end{tikzcd}$$

where $m : X \to Y$ is in $[\mathcal{Q}]$. Category $\mathcal{N}$ is initialized, hence there exists a morphism $r : Y \to A$ satisfying $r; h = q$. It remains only to prove $m; r = p$. The diagram above commutes, i.e. $m; q = p; h$. From $r; h = q$ we conclude $m; r; h = p; h$, and from $\mathcal{Q}$-bisimilarity being strong deterministic we obtain $m; r = p$, which completes the proof. □

Deterministic bisimilarity has an interesting property, that the closure of observation subcategory under colimits does not change induced bisimilarity. In Lemma 4.11 we show that $\mathcal{P}$-bisimilarity coincides with $\mathcal{P}^{\rightarrow}$-bisimilarity, where subcategory $\mathcal{P}^{\rightarrow}$ of $\mathcal{M}$ is defined as follows. It contains as objects colimits of all diagrams in $\mathcal{P}$. Let P and Q now be colimits of two diagrams D_P and D_Q of the same shape (i.e., D_P and D_Q are functors into P with a common domain). Let $\{p_i : P_i \to P\}_{i \in I}$ and $\{q_i : Q_i \to Q\}_{i \in I}$ denote colimit cocones of D_P and D_Q, respectively, where I denotes the class of objects of G. Every natural transformation $\xi : D_P \to D_Q$ induces a cocone $\{(\xi_i; q_i) : P_i \to Q\}_{i \in I}$ and results in the unique morphism $!_\xi : P \to Q$. Let category $\mathcal{P}^{\rightarrow}$ be the smallest subcategory of $\mathcal{M}$ containing all morphisms induced in this way by any two diagrams (over the same graph, i.e corresponding to functors from the same category) and any natural transformation (between these functors). In other words, morphisms of $\mathcal{P}^{\rightarrow}$ are all compositions $!_{\xi^1}; \ldots ; !_{\xi^k}$ of finite number of morphisms that can be obtained as described above.

Lemma 4.11 *For any subcategory $\mathcal{P}$ of $\mathcal{M}$ such that $\mathcal{P}$-bisimilarity is strongly deterministic, a morphism of $\mathcal{M}$ is $\mathcal{P}$-open iff it is $\mathcal{P}^{\rightarrow}$-open. Moreover, $\mathcal{P}^{\rightarrow}$-bisimilarity is also strongly deterministic.*

Proof: It is enough to prove that every $\mathcal{P}$-open morphism is also $\mathcal{P}^{\rightarrow}$-open, as $\mathcal{P}$ is a subcategory of $\mathcal{P}^{\rightarrow}$. Consider any commuting square:

$$\begin{tikzcd} P_0 \arrow[r, "\iota"] \arrow[d, "f"'] & A \arrow[d, "h"] \\ P_k \arrow[r, "\kappa"'] \arrow[ur, dashed, "r"] & B \end{tikzcd} \tag{7}$$

where h is $\mathcal{P}$-open, $f = !_{\xi^1}; \ldots ; !_{\xi^k}$ where $!_{\xi^i} : P_{i-1} \to P_i$ is the unique morphism induced by a natural transformation ξ^i and $P_0, \ldots, P_k$ are colimits of some diagrams in $\mathcal{P}$. We are going to construct a diagonal morphism r in diagram above.

It is enough to construct such a diagonal r for $k = 1$, i.e. for f = $!_\xi$ only ($\xi : P_0 \to P_k$) — the general case follows then by an easy induction w.r.t. k. Let D and D' denote the domain and codomain functor of the natural transformation $!_\xi$. D and D' are functors from the same category G such that P_0 and P_k are colimits of D and D', respectively. Let I denote the class of objects of G and let $\{p_i : D_i \to P_0\}_{i \in I}$ and $\{p'_i : D'_i \to P_k\}_{i \in I}$ be the colimit cocones. We define a family of morphisms $\{r_i : D'_i \to A\}_{i \in I}$ as diagonal morphisms in squares $\xi_i; (p'_i; \kappa) = (p_i; \iota); h$. As $\mathcal{P}$-bisimilarity is strongly deterministic, $\{r_i : D'_i \to A\}_{i \in I}$ is a cocone over D'. To show this, consider a morphism $m : D'_l \to D'_n$ in D', for some $l, n \in I$. We should show $r_l = m; r_n$. First observe that $r_l; h = m; r_n; h$, as $r_l; h = p'_l; \kappa = m; p'_n; \kappa = m; r_n; h$, since $r_n; h = p'_n; \kappa$. Now as $\mathcal{P}$ bisimilarity is strongly deterministic, $r_l; h = m; r_n; h$ implies $r_l = m; r_n$. Hence $\{r_i : D'_i \to A\}_{i \in I}$ is a cocone over D'. This gives us a unique morphism $r : P_k \to A$, making $p'_i; r = r_i$, for all $i \in I$. Now it is easy to verify that r is a diagonal morphism in diagram (7), by the universal property of colimits. Equality $r; h = \kappa$ follows from the fact that morphisms of a colimit cocone are "jointly epi". We have $p'_i; (r; h) = p'_i; \kappa$, for all $i \in I$ (since $p'_i; r = r_i$), hence $r; h = \kappa$. Similarly, to show that $!_\xi; r = \iota$, observe that $p_i; (!_\xi; r) = p_i; \iota$, for all $i \in I$, since morphisms $r_i = p'_i; r$ are diagonal and squares $\xi_i; p'_i = p_i; !_\xi$ commute from definition of $!_\xi$.

Strong determinism of $\mathcal{P}^{\rightarrow}$-bisimilarity follows easily from $\mathcal{P}$-bisimilarity being strongly deterministic and from the fact that morphisms of a colimit cocone are jointly epi. $\square$

4.2 Partial algebras

We are going to demonstrate that observational equivalence in the category of partial algebras can be defined as a bisimulation in the setting of open morphisms. We will actually characterize the observational equivalence twice: first (in this section) in the category $PAlg(\Sigma)$ of partial algebras with strong homomorphisms and then (in Section 5.1) in the richer category $PAlg_w(\Sigma)$ of weak partial homomorphisms. Restricting to only strong homomorphisms makes the task easier, so we start from that case, following the lines of [BT96], where observational equivalence for partial algebras was defined in category with strong morphisms.

The situation now is not so nice as in the case of standard algebras, where the forgetful functor has the left adjoint. Instead, there exists the left multiadjoint defined as follows. For any X in Set^{OBS}, the set $T(X) = \{T_i(X)\}_{i\in\mathcal{I}_X}$ consists of all partial algebras $T_i(X)$ whose carrier is a subset of the total algebra $T_\Sigma(X)$, which satisfy (for every $i \in \mathcal{I}_X$):

- $X \subseteq |T_i(X)| \subseteq |T_\Sigma(X)|$,
- carrier sets of $T_i(X)$ are closed under subterms, i.e. $t \in |T_i(X)|$ implies that all subterms of t are in $|T_i(X)|$,
- for any function symbol $f : s_1, \ldots, s_n \to s$ in Σ and terms $t_1, \ldots, t_n$ of appropriate sorts in $|T_i(X)|$, the function $f^{T_i(X)}$ is defined on $t_1, \ldots, t_n$ iff the term $f(t_1, \ldots, t_n)$ belongs to $|T_i(X)|$, and if so, then $f^{T_i(X)}(t_1, \ldots, t_n) = f(t_1, \ldots, t_n)$.

For any X in Set^{OBS}, we define the family $\eta_X = \{\eta_X^i : X \to |T_i(X)|_{OBS}\}_{i\in\mathcal{I}_X}$ of inclusions in Set^{OBS}.

Proposition 4.12 *The pair $\langle T = \{T(X)\}_{X\in|Set^{OBS}|}, \eta = \{\eta_X\}_{X\in|Set^{OBS}|}\rangle$ is the left multiadjoint to the functor $|\_|_{OBS}$.*

Having defined the multiadjoint, we easily obtain a conclusion from Lemmas 4.4, 4.5 and Theorem 4.3:

Corollary 4.13 *The observational equivalence of partial algebras in category $PAlg(\Sigma)$ coincides with the $\mathcal{Q}$-bisimilarity in $PAlg(\Sigma)$, for any subcategory $\mathcal{Q}$ of $[T(Set^{OBS})]$ containing $T(\mathbf{2}_{OBS})$; subcategory $\mathbf{2}_{OBS}$ of Set^{OBS} was defined in Section 3.1.*

Proof: The argument is essentially the same as for standard algebras in Corollary 3.6. □

5 Bisimilarity in pre-adjunction

In this section we study bisimilarity when the left multiadjoint to a functor $G : \mathcal{M} \to \mathcal{O}$ is not given. It is sufficient for our purposes to have a *pre-adjunction* (Definition 5.1) from which we extract a certain multiadjunction by distinguishing the class of *ext-strong* morphisms in $\mathcal{M}$. More precisely, left quaadjoint turns out to be the left multiadjoint to the functor G restricted to subcategory of ext-strong morphisms of $\mathcal{M}$. Then we compare the induced bisimilarities in both categories (Theorem 5.7) showing that they coincide for certain classes of observation objects. This coincidence gives us a tool for relating observational equivalences with bisimilarity. The results presented in this section are quite general; an illustrative example of application to partial algebras with weak homomorphisms can be found in Section 5.1.

Definition 5.1 (Left pre-adjoint, pre-adjunction) *Let $G : \mathcal{M} \to \mathcal{O}$ be a functor and X be any object of category $\mathcal{O}$. Let $F(X) = \{F_i(X)\}_{i \in \mathcal{I}_X}$ and $\eta_X = \{\eta^i_X : X \to G(F_i(X))\}_{i \in \mathcal{I}_X}$ be families of objects of $\mathcal{M}$ and morphisms of $\mathcal{O}$, respectively, both indexed by the same set $\mathcal{I}_X$. We say that a pair $\langle F(X), \eta_X\rangle$ is* a family of pre-free objects over X *(with* units η_X*) if*

(i) for any object A in $\mathcal{M}$ and a morphism $f : X \to G(A)$ there exists a (not necessarily unique) pair $\langle i, f' : F_i(X) \to A\rangle$ where $i \in \mathcal{I}_X$ and f' is a morphism in $\mathcal{M}$, such that $\eta^i_X;G(f') = f$ (we call such f' an extension *of f to A),*

(ii) $\eta^i_X;G(f') = \eta^i_X;G(f'')$ implies $f' = f''$, for any $f', f'' : F_i(X) \to A$ in $\mathcal{M}$, i.e. no two pairs with the same index i are allowed in (i),

(iii) for any object A in $\mathcal{M}$ and a morphism $f : X \to G(A)$, the set of indexes

$$\mathcal{I}^{f,A}_X = \{i \in \mathcal{I}_X \mid \text{ there exists } f' : F_i(X) \to A \text{ such that } \eta^i_X;G(f') = f\}$$

has the greatest element $\sqcup^{f,A}$ w.r.t. a partial order $\leq_X$ on $\mathcal{I}_X$ defined as:

$$i \leq_X j \quad \text{iff} \quad \text{there exists a morphism} \quad w : F_i(X) \to F_j(X) \quad \text{such that} \quad \eta^i_X;G(w) = \eta^j_X.$$

If for every object X in $\mathcal{O}$ there exists a family of pre-free objects $\langle F(X), \eta_X\rangle$ over X, we say that the pair of families $\langle F = \{F(X)\}_{X \in |\mathcal{O}|}, \eta = \{\eta_X\}_{X \in |\mathcal{O}|}\rangle$ is a left pre-adjoint *to G and that $\langle F, \eta\rangle$ and G form a* pre-adjunction.

Note Relation $\leq_X$ (for any X in $\mathcal{O}$) is a preorder, but it is easy to observe that $\leq_X$ is a partial order up to isomorphism, namely

$$i \leq_X j \text{ and } j \leq_X i \text{ iff } F_i(X) \simeq F_j(X),$$

for any $i, j \in \mathcal{I}_X$. Thus without loss of generality we can assume that $\leq_X$ is indeed a partial order.

From now on in this section we consider a fixed pre-adjunction formed by $\langle F, \eta\rangle$ and a functor $G : \mathcal{M} \to \mathcal{O}$. Let X, A be any objects in $\mathcal{O}$, $\mathcal{M}$, respectively. For any morphism $f : X \to G(A)$ in $\mathcal{O}$, by $f^\#$ we denote the unique morphism $f^\# : F_{\sqcup^{f,A}}(X) \to A$ such that $\eta^{\sqcup^{f,A}}_X;G(f^\#) = f$. We call $f^\#$ *the greatest extension* of f (to A).

Example As an example consider the category $PAlg_w(\Sigma)$ of partial algebras with weak homomorphisms. It turns out that the left multiadjoint $\langle T, \eta\rangle$, defined in Section 4.2 for category $PAlg(\Sigma)$, is now a left pre-adjoint to the functor $|\_|_{OBS} : PAlg_w(\Sigma) \to Set^{OBS}$ (cf. Section 5.1). The pre-free partial algebra $T_{\sqcup^{v,A}}(X)$ consists of precisely those terms over variables from X, which are defined in A under a valuation $v : X \to |A|_{OBS}$. Moreover, the greatest extension $v^\# : T_{\sqcup^{v,A}}(X) \to A$ is the unique extension of the valuation v to terms.

Now we are going to distinguish a subclass of *ext-strong* morphisms, giving raise to a multiadjunction.

Definition 5.2 *A morphism $f : A \to B$ in $\mathcal{M}$ is* ext-strong *if whenever composed with the greatest extension of a morphism $p : X \to G(A)$ to A gives the greatest extension of $(p;G(f))$ to B, i.e. if it satisfies the following*

$$p^\#;f = (p;G(f))^\#, \quad \text{for any morphism} \quad p : X \to G(A).$$

The condition above is equivalent to: $\sqcup^{p,A} = \sqcup^{(p;G(f)),B} \in \mathcal{I}_X$, for any X and $p : X \to G(A)$ in $\mathcal{O}$. The opposite implication holds for any morphism and we state this in a separate lemma for future references:

Lemma 5.3 *Let $h : A \to B$ in $\mathcal{M}$ and $p' : F_i(X) \to A$ be any morphisms in $\mathcal{M}$ being an extension of $p : X \to G(A)$. If $(p'; h)$ is the greatest extension (of $p; G(h)$) then p' is the greatest extension (of p).*

Proof: Assume that $(p'; h)$ is the greatest extension of $p; G(h)$ and at the same moment p' is not the greatest extension of p. Let $p'' : F_j(X) \to A$ be the greatest extension of p. But now $(p''; h)$ is also an extension of $(p; G(h))$ as well as $(p'; h)$. Moreover, $i <_X j$, which means that $(p'; h)$ cannot be the greatest. The contradiction completes the proof. □

Lemma 5.4 *Every ext-strong extension of a morphism f is the greatest extension of f.*

Proof: Let a morphism $f' : F_i(X) \to A$ in $\mathcal{M}$ be an extension of $f : X \to G(A)$. Assume f' is not the greatest. We will show f' is not ext-strong. Obviously $id_{F_i(X)}; f' = f'$. Moreover, $id_{F_i(X)} = (\eta^i_X)^\#$, i.e. is the greatest extension of η^i_X. Thus f' composed with the greatest extension gives f', which is not the greatest extension of $(\eta^i_X; G(f')) = f$, i.e. f' is not ext-strong morphism. □

The greatest extensions need not to be in general ext-strong and a simple counter-example can be constructed. But in many typical situations these two notions coincide (cf. Section 5.1), which justifies the following assumption, needed below for the results in this section:

Assumption 5.5 *Every greatest extension is ext-strong, i.e. for any A and $f : X \to G(A)$, $f^\#$ is ext-strong.*

It follows from Assumption 5.5 that the greatest extensions compose, since ext-strong morphisms compose. Moreover, identities (in fact, all isomorphisms) are ext-strong. Now, let $\mathcal{M}^{\text{ext-strong}}$ denote the subcategory of $\mathcal{M}$ consisting of all objects but only ext-strong morphisms.

Corollary 5.6 *The pair $\langle F, \eta \rangle$ is the left multiadjoint to G restricted to $\mathcal{M}^{ext\text{-}strong}$.*

Proof: Due to Assumption 5.5 all greatest extensions are in $\mathcal{M}^{\text{ext-strong}}$, hence for each object A in $\mathcal{M}$, every morphism $m : X \to G(A)$ in $\mathcal{O}$ has at least one extension to A, namely the greatest extension $m^\# : F_{\sqcup f,A}(X) \to A$. From Lemma 5.4 it follows that $m^\#$ is the only extension of m to A in $\mathcal{M}^{\text{ext-strong}}$. □

Let $\mathcal{P}$ be arbitrary fixed subcategory of $\mathcal{O}$. As a subcategory of observation objects in $\mathcal{M}$ we take similarly as in Section 4 the subcategory $F(\mathcal{P})$ whose objects are all pre-free objects and morphisms are all extensions of morphisms $(m; \eta^j_Y)$ for any morphism $m : X \to Y$ in $\mathcal{O}$ and any $j \in \mathcal{I}_Y$. We will deliberately use the same notation $F(\mathcal{P})$ for this subcategory of $\mathcal{M}$ as well as for different in general on morphisms subcategory $F(\mathcal{P})$ of $\mathcal{M}^{\text{ext-strong}}$ determined by the left multiadjoint (this subcategory was defined in Section 4).

In Theorem 5.7 we relate $F(\mathcal{P})$-openness in $\mathcal{M}$ and $F(\mathcal{P})$-openness in $\mathcal{M}^{\text{ext-strong}}$. It turns out that if all $F(\mathcal{P})$-open morphisms in $\mathcal{M}$ are ext-strong (i.e. are also in $\mathcal{M}^{\text{ext-strong}}$; obviously it is not always the case, for example for $\mathcal{P}$ being the empty category), then the notions of $F(\mathcal{P})$-openness coincide in both categories:

Theorem 5.7 *Let $\mathcal{P}$ be any subcategory of $\mathcal{O}$ such that all $F(\mathcal{P})$-open morphisms in $\mathcal{M}$ are ext-strong. Let $h : A \to B$ be a morphism in $\mathcal{M}$. Then h is $F(\mathcal{P})$-open in $\mathcal{M}$ iff h is $F(\mathcal{P})$-open in $\mathcal{M}^{ext\text{-}strong}$.*

The following easy lemma will be useful in the proof:

Lemma 5.8 *Let $f : F_i(X) \to A$ and $f' : F_j(X) \to A$ be two extensions of the same morphism, such that $i \leq_X j$, and let $s : F_i(X) \to F_j(X)$ satisfy $\eta^i_X; G(s) = \eta^j_X$ (existence of such a morphism is guaranteed by $i \leq_X j$). There holds:*

1. *$f = s; f'$,*
2. *s is determined uniquely (being an extension of the unit η^j_X) and is epi.*

(The observation that every morphism s witnessing the preorder $\leq_X$ is epimorphism, indicates that the pre-free objects over the same object X are closely related each to other.)

Proof: (of Theorem 5.7) $\Leftarrow$: To show $F(\mathcal{P})$-openness of h in $\mathcal{M}$ we need to find a diagonal morphism r in every commuting square:

$$\begin{tikzcd} F_i(X) \arrow[r, "p^i"] \arrow[d, "m"'] & A \arrow[d, "h"] \arrow[dl, dashed, "r"'] \\ F_j(Y) \arrow[r, "q^j"'] & B \end{tikzcd} \tag{8}$$

where m is an extension of $(n; \eta^j_Y)$, for some morphism $n : X \to Y$ in $\mathcal{P}$ (i.e. m is in $F(\mathcal{P})$), p^i, q^j are extensions of $p : X \to G(A)$ and $q : Y \to G(B)$, respectively. We construct another commuting square:

$$\begin{tikzcd} F_k(X) \arrow[r, "p^{\#}"] \arrow[d, "m'"'] & A \arrow[d, "h"] \arrow[dl, dashed, "r'"'] \\ F_l(Y) \arrow[r, "q^{\#}"'] & B \end{tikzcd} \tag{9}$$

where $k = \sqcup^{p,A} \in \mathcal{I}_X$, $l = \sqcup^{q,B} \in \mathcal{I}_Y$ (i.e. $p^{\#}$ and $q^{\#}$ are the greatest extensions of p and q, respectively). By Lemma 5.8(1) there exist morphisms $p' : F_i(X) \to F_k(X)$ and $q' : F_j(X) \to F_l(X)$ making

$$p'; p^{\#} = p^i \quad \text{and} \quad q'; q^{\#} = q^j. \tag{10}$$

Let m' be equal to the greatest extension of $(n; \eta^l_Y) = (n; \eta^j_Y; G(q'))$.

We should prove that both $p^{\#}$ and m' have the same domain objects, i.e.

$$k = \sqcup^{(n;\eta^l_Y), F_l(Y)}. \tag{11}$$

We know that $q^{\#}$ is ext-strong (by Assumption 5.5), thus $(m'; q^{\#})$ is the greatest extension of the morphism $(n; \eta^l_Y; G(q^{\#})) = (n; q)$. On the other hand $(p^{\#}; h)$ is the greatest extension (as h is ext-strong) of $(p; G(h))$. It is easy to observe that $p; G(h) = n; q$ — this follows from (10) commutativity of (8), thus both $(p^{\#}; h)$ and $(m'; q^{\#})$ are the greatest extensions of the same morphism. We conclude that $p^{\#}$ and m' have the same domain and moreover that square (9) commutes. This square consists of only ext-strong morphisms (by Assumption 5.5), thus from $F(\mathcal{P})$-openness of h in $\mathcal{M}^{\text{ext-strong}}$ we obtain a diagonal morphism r' in diagram (9). We will show that $r := q'; r'$ is diagonal in (8). Obviously, from (10) we conclude $r; h = q^j$, while equality $m; r = p^i$ follows from commutativity of

$$\begin{tikzcd} F_i(X) \arrow[r, "p'"] \arrow[d, "m"'] & F_k(X) \arrow[d, "m'"] \\ F_j(Y) \arrow[r, "q'"'] & F_l(Y) \end{tikzcd}$$

(this square commutes by Lemma 5.8(1) since $\eta^i_X; G(p') = \eta^k_X$ and and both m' and $(m; q')$ are extensions of $(n; \eta^l_Y))$.

$\Rightarrow$: To show $F(\mathcal{P})$-openness of h in $\mathcal{M}^{\text{ext-strong}}$ we need only to prove that in every commuting square (8) in $\mathcal{M}$ a diagonal morphism is ext-strong whenever p^i, q^j and m are ext-strong, i.e. (by Lemma 5.4) whenever p^i, q^j and m are greatest extensions. This follows from Lemma 5.3 (it is only necessary for q^j to be the greatest extension) and from Assumption 5.5. $\square$

The requirement needed in Theorem 5.7 is satisfied when $\mathcal{P} = \mathcal{O}$:

Lemma 5.9 *Every $F(\mathcal{O})$-open morphism in $\mathcal{M}$ is ext-strong.*

Proof: Let $h : A \to B$ be any $F(\mathcal{O})$-open morphism in $\mathcal{M}$, $p^\# : F_i(X) \to A$ be the greatest extension of $p : X \to G(A)$. We will show that $p^\#; h$ is the greatest extension of $(p; G(h))$, i.e. $i = j$, where $j := \sqcup^{(p;G(h)),B}$. It is easy to verify that $p^\#; h$ is an extension of $(p; G(h))$, i.e. that $i \leq_X j$. Let us consider the greatest extension $g : F_j(X) \to B$ of $(p; G(h))$. We will show $j \leq_X i$. Let $s : F_i(X) \to F_j(X)$ be a morphism meeting $\eta^i_X; G(s) = \eta^j_X$ (which exists as $i \leq_X j$). Consider a diagram:

$$\begin{tikzcd} F_i(X) \arrow[r, "p^\#"] \arrow[d, "s"'] & A \arrow[d, "h"] \arrow[dl, dashed, "r"'] \\ F_j(X) \arrow[r, "g"'] & B \end{tikzcd}$$

Obviously, s is in $F(\mathcal{P})$ as it is an extension of η^j_X. hence the square commutes by Lemma 5.8(1). By $F(\mathcal{O})$-openness of h there exists a diagonal morphism r. We will show that r extends p. Actually, $\eta^j_X; G(r) = \eta^i_X; G(s; r) = \eta^i_X; G(p^\#) = p$. This gives $j \leq_X i$ and completes the proof. $\square$

As a conclusion from Theorem 5.7 and Lemma 5.9 we obtain:

Corollary 5.10 *A morphism $h : A \to B$ in $\mathcal{M}$ is $F(\mathcal{O})$-open iff h is $F(\mathcal{O})$-open in $\mathcal{M}^{\text{ext-strong}}$.*

Determinism By Lemma 4.8 we conclude that all the bisimilarities considered so far, induced by an adjunction or a multiadjunction, are strongly deterministic. Here we prove that pre-adjunction also induces strongly deterministic bisimilarity if the related multiadjunction does so.

Lemma 5.11 *Let $\mathcal{P}$ be a subcategory of $\mathcal{O}$ as in Theorem 5.7. Whenever $F(\mathcal{P})$-bisimilarity in $\mathcal{M}^{\text{ext-strong}}$ is strongly deterministic then $F(\mathcal{P})$-bisimilarity in $\mathcal{M}$ is so as well.*

Proof: Assume $r'; h = s'; h$ for certain $F(\mathcal{P})$-open morphism h in $\mathcal{M}$. We will show $r' = s'$. Let r', s' have the domain, say, $F_i(X)$. We denote $r = \eta^i_X; G(r')$, $s = \eta^i_X; G(s')$. We have $r; G(h) = s; G(h)$, hence $(r; G(h))^\# = (s; G(h))^\#$. Due to the fact that h is ext-strong (as it is $F(\mathcal{P})$-open) we conclude $r^\#; h = (r; G(h))^\#$ and $s^\#; h = (s; G(h))^\#$, which gives us $r^\#; h = s^\#; h$. Morphism h is ext-strong and open in $\mathcal{M}^{\text{ext-strong}}$ by Theorem 5.7, so $r^\# = s^\#$ (as $F(\mathcal{P})$-bisimilarity in $\mathcal{M}^{\text{ext-strong}}$ is strongly deterministic), thus $r = s$. It means that $\eta^i_X; G(r') = \eta^i_X; G(s')$, from which we conclude $r' = s'$, as more that one extension is not allowed from one pre-free object. $\square$

Observation analogous to 4.6 holds also in this case, and can be concluded directly from the above lemma:

Corollary 5.12 *$F(\mathcal{O})$-bisimilarity is strongly deterministic for any left pre-adjoint F.*

5.1 Partial algebras with weak homomorphisms

The forgetful functor $|\_|_{OBS} : PAlg_w(\Sigma) \to Set^{OBS}$ has the left adjoint, but the full subcategory of free objects (in which all operations are totally undefined) is too small to induce the expected observational equivalence of partial algebras. In Section 4.2 we defined the left multiadjoint $\langle T, \eta\rangle$ to the functor $|\_|_{OBS}$. It turns out that the same pair is the left pre-adjoint when we work with weak morphisms:

Proposition 5.13 *Functor $|\_|_{OBS} : PAlg_w(\Sigma) \to Set^{OBS}$ together with the pair $\langle T, \eta\rangle$ form a pre-adjunction.*

The greatest extensions of this pre-adjunction are those homomorphisms which appeared in the multiadjunction as $v^{\#}$, for some $v : X \to |A|_{OBS}$ in Set^{OBS}. In the following we will need some weaker kind of strong morphisms. We say that $f : A \to B$ in $PAlg_w(\Sigma)$ is *R-strong* (for $R \subseteq sorts(\Sigma)$) if it reflects definedness of operations on the carrier of the subalgebra $\langle |A|_R\rangle_A$ of A generated by sorts R. The following lemma gives abstract characterization of OBS-strong partial homomorphisms:

Lemma 5.14 *A morphism $f : A \to B$ in $PAlg_w(\Sigma)$ is OBS-strong iff it is ext-strong.*

Proof: For any function $v : X \to |A|_{OBS}$ in Set^{OBS} let T^v_A denote the set of terms over X defined in A under the valuation v, or equivalently, the sum of carrier sets of all pre-free algebras over X from which there exists an extension of v. Consider any morphism $h : A \to B$ in $PAlg_w(\Sigma)$. Now we have an easy characterizations of both notions by means of sets T^v_A and $T_B^{(v;|h|_{OBS})}$: morphism h is ext-strong (respectively, OBS-strong) iff

$$T^v_A = T_B^{(v;|h|_{OBS})}, \text{ for any } \; v : X \to |A|_{OBS} \text{ in } \; Set^{OBS}.$$

□

Assumption 5.5 is satisfied in this case:

Lemma 5.15 *Every greatest extension in $PAlg_w(\Sigma)$ w.r.t. forgetful functor $|\_|_{OBS}$ is OBS-strong.*

Proof: For every extension $v^i : T_i(X) \to A$ which is not OBS-strong we can construct another ("greater") extension $v^j : T_j(X) \to A$, such that the carrier set of $T_j(X)$ is a superset of $|T_i(X)|$. For example $|T_j(X)|$ can contain precisely one element more, being a term of application of an operation of Σ which is defined in A on arguments being values of some terms from $|T_i(X)|$ under the valuation $\eta^i_X; |v^i|_{OBS}$. Such an operation and terms exist since v^i is not OBS-strong. This implies that v^i is not the greatest extension. □

Category $PAlg_w(\Sigma)^{ext-strong}$ contains more morphisms than $PAlg(\Sigma)$, namely all OBS-strong partial homomorphisms. Now, by Corollary 5.10, the class of $T(Set^{OBS})$-open morphisms in category $PAlg_w(\Sigma)$ is equal to the class of $T(Set^{OBS})$-open morphisms in $PAlg_w(\Sigma)^{ext-strong}$. The subtle point here is that the left multiadjoint to $|\_|_{OBS}$ for category $PAlg(\Sigma)$ works also for $PAlg_w(\Sigma)^{ext-strong}$, since every OBS-strong extension is a strong homomorphism. This observation implies that $T(Set^{OBS})$-bisimilarity in $PAlg_w(\Sigma)$ coincides with $T(Set^{OBS})$-bisimilarity in $PAlg(\Sigma)$, from which we conclude Corollary 5.16. It gives a characterization of observational equivalence of partial algebras in terms of open morphisms in the category of weak partial homomorphisms:

Corollary 5.16 *The observational equivalence of partial algebras coincides with the $T(Set^{OBS})$-bisimilarity in the category $PAlg_w(\Sigma)$.*

Proof: By direct application of Corollary 5.10, $T(Set^{OBS})$-bisimilarities in categories $PAlg_w(\Sigma)$ and $PAlg_w(\Sigma)^{ext-strong}$ coincide. Moreover, by Corollary 4.13, observational equivalence of partial algebras coincides with $T(Set^{OBS})$-bisimilarity in $PAlg(\Sigma)$. Thus we have only to prove:

$$A \sim_{T(Set^{OBS})} B \text{ in } PAlg(\Sigma) \iff A \sim_{T(Set^{OBS})} B \text{ in } PAlg_w(\Sigma)^{ext-strong},$$

for any partial algebras A and B. The implication from left to right is trivial, as strong partial homomorphisms are OBS-strong. To show the opposite implication, observe that when there exists a span $A \xleftarrow{f} C \xrightarrow{g} B$ of observational OBS-strong morphisms, then morphisms $(\iota_{\hookrightarrow C}; f)$ and $(\iota_{\hookrightarrow C}; g)$ are strong, where $\iota_{\hookrightarrow C} : \langle |C|_{OBS} \rangle_C \to C$ is the canonical inclusion. This means that the span $A \xleftarrow{\iota_{\hookrightarrow C};f} C \xrightarrow{\iota_{\hookrightarrow C};g} B$ is in $PAlg(\Sigma)$, and since morphisms $(\iota_{\hookrightarrow C}; f)$ and $(\iota_{\hookrightarrow C}; g)$ are obviously observational, we conclude that A and B are $T(Set^{OBS})$-bisimilar in $PAlg(\Sigma)$, which completes the argument. □

We are able to restrict the subcategory $T(Set^{OBS})$ of observation objects from Corollary 5.16. Define $\mathbf{Fin}_{OBS}$ as the full subcategory of $T(Set^{OBS})$ consisting of algebras with finite carrier sets. Note that $\mathbf{Fin}_{OBS}$ is not a full subcategory of $PAlg_w(\Sigma)$ in general. We prove that observational equivalence of partial algebras is equal to $\mathbf{Fin}_{OBS}$-bisimilarity in $PAlg_w(\Sigma)$.

Lemma 5.17 *A (weak) partial homomorphism is $T(Set^{OBS})$-open iff it is $\mathbf{Fin}_{OBS}$-open.*

Proof: As $\mathbf{Fin}_{OBS}$ is a subcategory of $T(Set^{OBS})$, we have only to prove that $\mathbf{Fin}_{OBS}$-openness implies $T(Set^{OBS})$-openness, i.e., by Corollary 5.10, that every $\mathbf{Fin}_{OBS}$-open partial homomorphism is *i)* bijective on OBS (when $OBS \neq \emptyset$) and *ii)* OBS-strong. Let h be a $\mathbf{Fin}_{OBS}$-open partial homomorphism.

For any finite set Γ of terms closed under subterms, by $P(\Gamma)$ we denote the pre-free partial algebra (over the set V of variables in Γ) in $\mathbf{Fin}_{OBS}$ whose carrier set is Γ. Recall from Section 4.2 that operations in $P(\Gamma)$ are defined canonically as in the term algebra $T_\Sigma(V)$ of undefined.

i) Let $s \in OBS$ be a fixed observable sort. We ignore the carriers of pre-free algebras of other sorts, assume they are empty sets. Assume that $|B|_s$ is non-empty.

Surjectivity on the carrier of sort s follows from the openness condition for the right-bottom triangle of the commuting diagram (we draw only carrier sets of sort s of pre-free algebras):

$$\begin{tikzcd} P(\emptyset) \arrow[r, "i_A"] \arrow[d, "i_{P(\{x\})}"'] & A \arrow[d, "h"] \\ P(\{x\}) \arrow[r, "q"'] \arrow[ur, "r"] & B \end{tikzcd}$$

(x is a variable of a sort s, i_X is the unique homomorphism from the initial partial algebra $P(\emptyset)$ to X), as q can point to any element of $|B|_s$.

To show injectivity of $|h|_s$, assume $h(a_1) = h(a_2)$, for $a_1, a_2 \in |A|_s$. Consider the commuting diagram:

$$\begin{tikzcd} P(\{x_1, x_2\}) \arrow[r, "p"] \arrow[d, "m"'] & A \arrow[d, "h"] \\ P(\{y\}) \arrow[r, "q"'] \arrow[ur, "r"] & B \end{tikzcd}$$

where $p(x_i) = a_i$, $m(x_i) = y$ $(i = 1, 2)$, $q(y) = h(a_1)$, x_1, x_2, y are variables of sort s. Any morphism $r : P(\{y\}) \to A$ making the upper-left triangle commute satisfies $|r|_s(y) = a_i$, so $a_1 = a_2$.

ii) For any $f : s_1, \ldots, s_n \to s$ in Σ, $a_1 \in A_{s_1}, \ldots, a_n \in A_{s_n}$, such that $a_i \in \langle |A|_{OBS} \rangle_A$, $i = 1, \ldots, n$, let us assume that f^B is defined on $h(a_1), \ldots, h(a_n)$. For any a_i, there exist a term t^i with variables only of observable sorts, and a valuation $v : V \to A$ such that ${t^i}_{v[A]} = a_i$, where $V = \bigcup_i var(t^i)$. We can assume that sets $var(t^i)$ are distinct, for $i = 1, \ldots, n$, hence there exists valuation v making ${t^i}_{v[A]}$ $= a_i$, for $i = 1, \ldots, n$. We define the commuting diagram as follows.

$$\begin{tikzcd} P(\Gamma) \arrow[r, "p"] \arrow[d, "m"'] & A \arrow[d, "h"] \\ P(\Omega) \arrow[r, "q"'] \arrow[ur, "r"] & B \end{tikzcd}$$

Let Γ be a closure under subterms of $\{t_1, \ldots, t_n\}$, $\Omega = \Gamma \cup \{f(t_1, \ldots, t_n)\}$, and $m : P(\Gamma) \to P(\Omega)$ be the canonical inclusion. Morphisms p and q are extensions of v to A and B, respectively. Any diagonal morphism $r : P(\Omega) \to A$ (making the bottom-right triangle commute) transforms the term $f(t^1, \ldots, t^n)$ to $f^A(a_1, \ldots, a_n)$, thus f^A is defined on $a_1, \ldots, a_n$. $\square$

The subcategory $\mathbf{Fin}_{OBS}$ of finite partial pre-free algebras plays in the category $PAlg_w(\Sigma)$ a role of finite computations, analogously as the subcategory $Bran_L$ of finite linear transition systems in the category of transition systems [JNW93]. Let us summarize the results concerning open bisimilarity in category $PAlg_w(\Sigma)$, as a direct conclusion from Corollary 5.16, Lemma 5.17, Corollary 5.12 and Lemma 4.10:

Corollary 5.18 *The observational equivalence of partial algebras coincides with the $\mathcal{Q}$-bisimilarity in $PAlg_w(\Sigma)$, for any subcategory $\mathcal{Q}$ of $[T(Set^{OBS})]$ containing* $\mathbf{Fin}_{OBS}$.

(The coincidence of $T(Set^{OBS})$- and $[T(Set^{OBS})]$-bisimilarity follows from the fact that first one is deterministic.)

6 Final remarks

The main results of this paper are characterizations of observational equivalences for a number of classes of algebras as the bisimilarity induced by a subcategory of free, multifree or pre-free objects. We developed the framework of pre-adjunction and stated some facts for multiadjunction and pre-adjunction, which enabled us to show the coincidence of observational equivalence and bisimilarity. We identified small observation subcategories, corresponding to "finitary" observations, which are sufficient to induce the observational equivalences. This can be very useful in proving observational equivalence of algebras. Moreover, we observed that the bisimilarities considered in the paper are deterministic, in contrast to bisimilarities between model for concurrent systems like transition systems or event structures.

As a response to numerous definitions of observational equivalence for different categories of algebras, the idea of behavioural equivalence was generalized in [BT96] to an arbitrary *concrete category*, i.e. assumed to be equipped with a faithful concretization functor to Set^S. The interesting continuation of the work is to try to apply the machinery of the pre-adjunction in such a general case. Moreover, some work is needed to examine applicability of our results to other frameworks than considered in this paper, for example to continuous algebras.

There arises also another interesting issue: is it possible, for an arbitrary bisimilarity induced by some class of open maps in a concrete category, to find partial congruences factorizing this bisimilarity? We have found a partial answer to this question [Las97], but elegant necessary and sufficient conditions for such a coincidence are yet to be developed.

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