Partial-Congruence Factorization of Bisimilarity Induced by Open Maps*

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Abstract

We investigate relationships between bisimilarity defined by open maps and behavioural equivalence *factorized* by indistinguishability relations. This is done in the setting of a concrete category supporting algebraic constructions of subobject and quotient. Under some mild assumptions, a sufficient condition is found for bisimilarity to be factorizable by the greatest *open congruences*. Moreover, we find a necessary and sufficient condition for a factorization of bisimilarity: it is factorizable iff quotients by all maximal open congruences are isomorphic. The general results are motivated and illustrated by important examples: transition systems, event structures and presheaf models.

Keywords. Behavioural equivalence, bisimilarity, open maps, models for concurrency.

Introduction

The concept of *a behavioural equivalence* is a fundamental notion in programming methodology, as it seems to capture appropriately the "black box" character of data abstraction. This general notion instantiates to a behavioural equivalence of sequential programs or data types [Rei81, GGM76], as well as to the bisimulation equivalence (bisimilarity) of concurrent processes [Par81, Miln89].

Recently a categorical generalization of bisimulation was proposed, by means of *open maps* (*open morphisms*) [JNW93], enabling a uniform definition of bisimulation equivalence across a range of different models for parallel computations. This setting turned out appropriate for defining, among others, strong and weak bisimilarity [Miln89], trace equivalence, testing equivalence and strong history preserving bisimulation of event structures (see [CN95], [JNW93] for an overview). Open maps can be understood as arrows witnessing a bisimulation, hence two objects A and B in a category are bisimilar if they are related by a span of open maps:

$$\begin{tikzcd} & C \arrow[dl] \arrow[dr] & \\ A & & B \end{tikzcd}$$

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representing abstractly a bisimulation.

A dual approach was used for defining the behavioural equivalence for algebras. Roughly speaking, two algebras A and B are *behaviourally equivalent* if their quotients by *indistinguishability relations* $\sim_A$ and $\sim_B$ are isomorphic [BHW95], that is if they are related by a co-span of quotient projections:

$$\begin{tikzcd} A \arrow[dr] & & B \arrow[dl] \\ & A/_{\sim_A} \cong B/_{\sim_B} & \end{tikzcd}$$

We say that behavioural equivalence is *factorized* by the family of indistinguishability relations. This scheme is appropriate for various categories of algebras, and more generally for an arbitrary *concrete category* supporting algebraic constructions of subobject and quotient [BT96].

The undeniable conceptual analogies (despite technical differences) between both approaches are well recognized and much work was done in linking the two notions, mainly in applying methods based on bisimulation to sequential programs [AO93, Gor95]. In this paper, roughly speaking, we investigate the opposite direction, namely we show that algebraic techniques are applicable to bisimulation of processes as well. This is a completion of [Las97], where the author (re-)defined observational equivalences for standard, partial and regular algebras ([BT95]) by means of open maps.

The main problem investigated in this paper is to find a family of relations factorizing open-map bisimilarity. Our motivation was the observation that for strong bisimilarity of transition systems such a factorization exists and is given by the greatest bisimulations (Section 2). In Section 4 we present a successful attempt to imitate these properties in an arbitrary concrete category. We find general conditions sufficient, under some mild assumptions, for the greatest congruences induced by open maps (the greatest *open congruences*) to factorize bisimilarity. Most interestingly, these very general results apply to some important examples: transition systems (which fit into the general scheme) and presheaf models [JM94], proposed in [JNW93] as a uniform model for concurrency.

In some relevant cases the greatest open congruences need not exist, for instance in the category of event structures [NW95]. In Section 5 we prove that for a factorization it suffices to have a weaker property, that is the quotients by all maximal open congruences to be isomorphic. Moreover, this condition turns out to be necessary, which gives an elegant necessary and sufficient condition for a factorization of a wide class of bisimilarities defined by open maps. The crucial example is the subcategory of *simple* event structures (Section 6), failing to have greatest open congruences, but satisfying the weaker requirement.

This paper is not the only attempt to define bisimilarity by means of co-spans. An observation central to [C85, BB88] is that two transition systems are strongly bisimilar precisely when they are related by a co-span of open maps (zig-zag maps). Our work is hence a generalization of this development, specific for transition systems. Moreover, we give a response to the negative statement from [JNW93] that the "open co-span" approach can not be applied to the bisimilarity of *transition systems with independence* [NW95], general-Ising both transition systems and event structures. Our results on event structures give some bound for extendibility of this negative statement.

This paper is an extended and revised version of [Las98]. After some preliminary definitions in Section 1, we start in Section 2 with some intuition based on strong bisimilarity between transition systems. This should be a preparation before Sections 3, 4 and 5, being more general and abstract, hence unavoidably less understandable.

1 Preliminaries

Transition systems

Transition systems are widely used as models for concurrent systems (cf. [NW95]). Throughout the paper let L be a fixed set of labels. A *labelled transition system* (over L) is a structure $T = (S, i, t)$, where S is a set of *states* with an initial state $i \in S$ and $t \subseteq S \times L \times S$ is a *transition relation*. We write $s \xrightarrow{a} s'$ for $(s, a, s') \in t$. A morphism between two transition systems $T_1 = (S_1, i_1, t_1)$, $T_2 = (S_2, i_2, t_2)$, is a function $f : S_1 \to S_2$, such that $f(i_1) = i_2$ and $s \xrightarrow{a} s'$ implies $f(s) \xrightarrow{a} f(s')$. This defines a category TS_L (morphisms compose as functions). Below we adopt the definition of [Miln89] to our setting of transition systems with initial states:

Definition 1.1 *A* strong bisimulation *between T_1 and T_2 is any relation $\sim \subseteq S_1 \times S_2$ such that $(i_1, i_2) \in \sim$, and whenever $s_1 \sim s_2$,*

- *$s_1 \xrightarrow{a} s_1'$ in T_1 implies $s_2 \xrightarrow{a} s_2'$ in T_2 and $s_1' \sim s_2'$, for some s_2',*
- *$s_2 \xrightarrow{a} s_2'$ in T_2 implies $s_1 \xrightarrow{a} s_1'$ in T_1 and $s_1' \sim s_2'$, for some s_1'.*

We say that T_1 and T_2 are strongly bisimilar *if there exists a strong bisimulation between them.*

Event structures

The following definitions are cited from [JNW93] and [NW95] and specialised for a fixed set of labels. A *labelled event structure* (over L) is a structure $E = \langle |E|, \leq_E, Con_E, l_E \rangle$ consisting of a set $|E|$ of events partially ordered by $\leq_E$, the *causal dependency relation*, a family $Con_E \subseteq \mathcal{P}^{fin}(|E|)$ of finite subsets of events (the *consistency relation*), and a labelling function $l_E : |E| \to L$ satisfying:

- $\lceil e \rceil = \{e' : e' \leq_E e\}$ is finite, for every $e \in |E|$,
- $\{e\} \in Con_E$,
- $Y \subseteq X \in Con_E \Rightarrow Y \in Con_E$,
- $X \in Con_E \wedge e \leq_E e' \in X \Rightarrow X \cup \{e\} \in Con_E$.

A *configuration* of E, representing a state of computation, is any subset C of events being:

- *downward closed*: $e' \in C \wedge e \leq_E e' \Rightarrow e \in C$,
- *consistent*: $\forall X \subseteq^{fin} C \,.\, X \in Con_E$.

A morphism of event structures $h : E \to F$ is a function $h : |E| \to |F|$ satisfying:

- $l_F(h(e)) = l_E(e)$, for any $e \in |E_1|$ (preservation of labels),
- if C is a configuration of E then $h(C)$ is a configuration of F and h restricted to C is 1-1.

Event structures with their morphisms form a category ES_L. For configurations C, C' of E and any $a \in L$, by $C \xrightarrow{a} C'$ we mean $e \notin C, C' = C \cup \{e\}$ and $l_E(e) = a$. Note that morphisms of event structures preserve transitions, namely $C \xrightarrow{a} C'$ in E implies $h(C) \xrightarrow{a} h(C')$ in F.

Definition 1.2 *A* history preserving bisimulation *(hp-bisimulation in short) between event structures E_1, E_2 is a set H of triples (C_1, ξ, C_2), where C_1 is a finite configuration of E_1, C_2 is a finite*[1] *configuration of E_2 and ξ is an isomorphism of pomsets between them, such that $(\emptyset, \emptyset, \emptyset) \in H$, and whenever $(C_1, \xi, C_2) \in H$,*

- *if $C_1 \xrightarrow{a} C_1'$ in E_1 then $C_2 \xrightarrow{a} C_2'$ in E_2 and $(C_1', \xi', C_2') \in H$, with $\xi \subseteq \xi'$, for some C_2', ξ',*
- *if $C_2 \xrightarrow{a} C_2'$ in E_2 then $C_1 \xrightarrow{a} C_1'$ in E_1 and $(C_1', \xi', C_2') \in H$, with $\xi \subseteq \xi'$, for some C_1', ξ'.*

(By C_1' and C_2' we denote finite configurations of E_1 and E_2, respectively.) A hp-bisimulation H is strong *(shp-bisimulation in short) when it further satisfies*

- *$(C_1, \xi, C_2) \in H$ and $C_1' \subseteq C_1$ implies $(C_1', \xi', C_2') \in H$, for some $\xi' \subseteq \xi$ and $C_2' \subseteq C_2$,*
- *$(C_1, \xi, C_2) \in H$ and $C_2' \subseteq C_2$ implies $(C_1', \xi', C_2') \in H$, for some $\xi' \subseteq \xi$ and $C_1' \subseteq C_1$.*

Event structures E_1, E_2 are shp-bisimilar *if there exists a shp-bisimulation between them.*

History preserving bisimilarity was invented by many authors independently, among others in [RT88] and [vGG89]. The shp-bisimilarity has been first introduced in [Bed91] (called there *hereditary* history preserving bisimilarity) and rediscovered independently in [JNW93]; similar ideas can be found already in [DNM89].

Behavioural equivalence in a concrete category

We introduce here a categorical setting, borrowed from [BT96], in which we are going to work in the following sections. All the definitions below, being very abstract, follow usual algebraic intuitions. We aim at identifying abstractly some constructions on algebras, like quotients and subobjects, sufficient for dealing with behavioural equivalence.

Let $\mathbb{M}$ be an arbitrary category, equipped with a concretization functor $|\_| : \mathbb{M} \to Set^S$ to the category of sets indexed by a set S (category of S-sorted sets and functions), which we require to be faithful (although in most of examples S will be a singleton set). For an object A, $|A|$ can be seen as a carrier of A, hence we can think of objects of $\mathbb{M}$ as "sets with structure". Moreover, faithfulness of $|\_|$ allows us to treat morphisms as (structure preserving) functions between carrier sets.

For instance, category TS_L of transition systems over a fixed labelling set L can be equipped with the forgetful functor $|\_| : TS_L \to Set$, defined on objects by $|(S, i, t)| = S$ and on morphisms by $|f| = f$. A natural choice for a concretization functor for category ES_L of events structures is $|\_| : ES_L \to Set$, taking an event structure E to the set $|E|$ of its events and a morphism h to its underlying function $|h|$.

Subobjects In the following we use a (deliberately overloaded) notation $|\mathbb{M}|$ for the class of objects of $\mathbb{M}$. $A \in |\mathbb{M}|$, a *subobject* of A is an object B together with a morphism $\iota_{B \hookrightarrow A} : B \to A$ such that $|\iota_{B \hookrightarrow A}|$ is an inclusion. A subobject B is *full* if for any morphism $f : C \to A$ with $|f|(|C|) \subseteq |B|$, there exists a (necessarily unique) morphism $\widehat{f} : C \to B$ satisfying $\widehat{f}; \iota_{B \hookrightarrow A} = f$. Intuitively speaking, the full subobject inherits the whole "structure" of A restricted to $|B|$. Given an object A and a set $X \subseteq |A|$, a *full subobject of A generated by X*, denoted by $\langle X \rangle_A$, is a full subobject B of A such that $X \subseteq |B|$ and such that for any other subobject C of A

[1] In [JNW93] infinite configurations are also allowed in bisimulations. We restrict here ourselves to finite ones, following [vGG89], as this has no influence on induced bisimilarity relation.

either $|B| \subseteq |C|$ or $X \not\subseteq |C|$. $\langle X\rangle_A$, if exists, is determined up to *an exact isomorphism* (that is an isomorphism i with $|i|$ being identity). $\mathbb{M}$ *has full subobjects* if for each object A and a set $X \subseteq |A|$, there exists a full subobject generated by X.

Congruences and quotients For any morphism $f : A \to B$ in $\mathbb{M}$, by a *kernel* of f we mean an equivalence $\{(a, a') \in |A| \times |A| : |f|(a) = |f|(a')\}$, denoted by $\ker(|f|)$. Given an object A in $\mathbb{M}$, by a *congruence* on A we mean the kernel $\ker(|f|)$ of some morphism $f : A \to B$.

A *quotient* of an object A is an object B together with an epimorphism $\pi : A \to B$, called the *quotient projection*, such that for any morphism $f : A \to C$ with $\ker(|\pi|) \subseteq \ker(|f|)$ there exists a (necessarily unique) morphism $\bar{f} : B \to C$ satisfying $\pi; \bar{f} = f$. For a congruence $\sim$ on A, a quotient of A by $\sim$ is a quotient B of A whose projection satisfies $\ker(|\pi|) = \sim$. Given $\sim$, this quotient projection will be also denoted by $\pi_{A/\sim}$, and B will be written as $A/\sim$. Quotients by congruences, unlike generated subobjects, are determined up to isomorphism. We say that $\mathbb{M}$ *has surjective quotients* if for any A and any congruence $\sim$ on A, the quotient $A/\sim$ exists and the projection $|\pi_{A/\sim}|$ is surjective.

Indistinguishability structures By a *partial congruence* on an object A we mean a congruence on a full subobject of A, called its *domain subobject.* In the sequel we often deal with a family of partial congruences consisting, for each object A in $\mathbb{M}$, of a partial congruence $\simeq_A \subseteq |A| \times |A|$. We will write $\simeq$ for $\{\simeq_A\}_{A\in|\mathbb{M}|}$, and then, for each object A, we will write $A_\simeq$ for a domain subobject of $\simeq_A$. When no danger of confusion arises we will omit subscripts and write $A/\simeq$ instead of $A_\simeq/\simeq_A$, $\pi_{A/\simeq}$ instead of $\pi_{A_\simeq/\simeq_A}$, etc. Such a family $\simeq$ we call an *indistinguishability structure*, since $\simeq_A$ is meant to represent a (behavioural) indistinguishability in A. We say that two objects A and B are *behaviourally equivalent* (w.r.t. $\simeq$), which we denote by $A \equiv_\simeq B$, if $A/\simeq$ and $B/\simeq$ are isomorphic.

For a justification of this abstract setting we recall here definition of observational equivalence of standard many-sorted algebras over a signature Σ. Let us fix a subset OBS of *observable sorts* in Σ. By *observational morphisms* we mean those Σ-homomorphisms which are bijective on observable sorts.

For a Σ-algebra A, we define the partial congruence $\sim_A^{OBS}$ as follows. The domain of $\sim_A^{OBS}$ is the carrier of the subalgebra of A generated by the carrier of observable sorts. This is the *smallest* observational subalgebra, i.e., the smallest subalgebra with observational inclusion. Then we define $\sim_A^{OBS}$ as the greatest congruence on that subalgebra, which is identity on observable sorts. In fact, this is the *greatest* observational congruence, i.e., the greatest congruence with observational quotient projection.

Intuitively, the partial equivalence $\sim_A^{OBS}$ is intended to be a relation of *indistinguishability* of elements of A. The observational equivalence of Σ-algebras is an equivalence relation $\equiv_{OBS}$ *factorized* by the family$\{\sim_A^{OBS}\}_{A\in|Alg_\Sigma|}$ of partial congruences, i.e., defined by:

$$A \equiv_{OBS} B \quad \text{iff} \quad A/_{\sim_A^{OBS}} \cong B/_{\sim_B^{OBS}}.$$

(The quotient $A/_{\sim_A^{OBS}}$ of A by the partial congruence $\sim_A^{OBS}$ denotes the quotient of the domain subobject by $\sim_A^{OBS}$). For further details we refer to [BT96]; interestingly, as noticed there, in the way sketched above one can uniformly introduce observational equivalence in other algebraic settings, e.g. for partial and regular algebras.

There exist an explicit characterization of $\sim_A^{OBS}$ as the relation of *indistinguishability* by means of observable Σ-contexts ([BHW95]). Similar characterizations for partial and regular algebras can be found in [BT96] and [BT95], respectively.

Bisimulation from open maps

Let $\mathbb{M}$ be a category of models of computation, in which we choose a subcategory $\mathbb{P}$ (not necessarily full) of *observation objects*. Any morphism $p : O \to A$ from an observation object $O \in |\mathbb{P}|$ is understood as an observable *computation* in A. A morphism $h : A \to B$ between models can be intuitively thought of as a simulation of A in B since h transforms every computation $p : O \to A$ in A to a computation $p; h : O \to B$ in B. Moreover any morphism $m : O \to O'$ in $\mathbb{P}$ making $p = m; p'$ means intuitively that a "larger" computation p' is an extension of p (via m).

Definition 1.3 (Open maps and bisimilarity) *A morphism $h : A \to B$ in $\mathbb{M}$ is* $\mathbb{P}$-open *if for any morphism $m : O \to O'$ in $\mathbb{P}$ and $p : O \to A$ and $p' : O' \to B$ in $\mathbb{M}$, whenever the square*

$$\begin{tikzcd} O \arrow[r, "p"] \arrow[d, "m"'] & A \arrow[d, "h"] \\ O' \arrow[r, "p'"'] \arrow[ur, "r"] & B \end{tikzcd} \tag{1}$$

commutes, i.e. $p; h = m; p'$, there exists a diagonal *morphism $r : O' \to A$ in $\mathbb{M}$ making two triangles commute, i.e. $p = m; r$ and $p' = r; h$. Two models A and B are* $\mathbb{P}$-bisimilar*, denoted by $A \sim_{\mathbb{P}} B$, if there exists in $\mathbb{M}$ a span of $\mathbb{P}$-open maps: $A \longleftarrow C \longrightarrow B$.*

We omit the prefix $\mathbb{P}$- when it is obvious from a context. The notion of $\mathbb{P}$-bisimilarity generalises the strong bisimilarity between transition systems, which was shown in [JNW93] to coincide with $Bran_L$-bisimilarity, where $Bran_L$ is the full subcategory of transition systems consisting of finite sequences of actions:

$$i \xrightarrow{a_1} s_1 \xrightarrow{a_2} \dots \xrightarrow{a_n} s_n \tag{2}$$

$Bran_L$-open morphisms $f : (S_1, i_1, t_1) \to (S_2, i_2, t_2)$ are those satisfying the following *zig-zag* property: for each reachable $s \in S_1$, whenever $f(s) \xrightarrow{a} s'$, then $s \xrightarrow{a} s''$, for some $s'' \in S_1$ satisfying $f(s'') = s'$ (they are also called *zig-zag morphisms*); a state s of a transition system (S, i, t) is *reachable* if some finite sequence of transitions (2) ends in s, i.e. $s_n = s$.

To obtain a shp-bisimilarity of event structures we need some branching structure in observation objects: shp-bisimilarity coincides with Pom_L-bisimilarity [JNW93], where Pom_L is the full subcategory of ES_L consisting of finite *pomsets*, i.e. finite event structures with consistency relation containing all finite subsets of events. A morphism $f : E \to F$ is Pom_L-open iff

1. f satisfies the "zig-zag" property: for any finite configuration C of E, if $f(C) \xrightarrow{a} C'$ in F then $C \xrightarrow{a} C''$ in E and $f(C') = C''$, for some configuration C' of E.

2. f reflects concurrency: for all different events $e_1, e_2 \in |E|$, such that $\{e_1, e_2\} \in Con_E$,

$$f(e_1)\ co_F\ f(e_2) \Rightarrow e_1\ co_E\ e_2,$$

where $e_1\ co_E\ e_2$ means that events are *concurrent*, i.e. consistent, but causally not related: $\{e_1, e_2\} \in Con_E$, $e_1 \not\leq e_2$ and $e_2 \not\leq e_1$.

Note that the implication opposite to 2 holds for every (not necessarily open) morphism of event structures: $e_1\ co_E\ e_2 \Longrightarrow f(e_1)\ co_F\ f(e_2)$.

2 Motivating example

We start in this section with some intuition based on strong bisimilarity between transition systems. This should be a preparation before Sections 3, 4 and 5, being more general and abstract, hence unavoidably less understandable.

Let us focus for a while on the category of transition systems, and try to find out an indistinguishability structure $\doteq = \{\doteq_T\}_{T\in|TS_L|}$ factorizing strong bisimilarity. Transition systems are intended to serve as an intuitive example and a justification for the considerations in the following sections. For a domain subobject of a transition system T it is reasonable to choose its part "observable" by means of computations; the good candidate is the subobject consisting of all reachable states of T and inheriting all the transitions between reachable states. Observe that $T_{\doteq}$ defined in this way is precisely the full subobject generated by the union of all images of computations in T (compare with (5) in Section 3).

As the indistinguishability relation $\doteq_T$ on $T_{\doteq}$ take the greatest bisimulation between $T_{\doteq}$ and $T_{\doteq}$ itself. The following lemma justifies this choice and implies that $\doteq_T$ is actually a congruence.

Lemma 2.1 *An equivalence $\sim\ \subseteq |T_{\doteq}| \times |T_{\doteq}|$ is a bisimulation iff it is a kernel of a zig-zag morphism from $T_{\doteq}$.*

Proof: $\Leftarrow$: It is enough to show, for any zig-zag morphism $f : T_{\doteq} \to U$, that $f(s_1) = f(s_2)$ and $s_1 \xrightarrow{a} s_1'$ in T implies $s_2 \xrightarrow{a} s_2'$ in T and $f(s_1') = f(s_2')$, for certain s_2'. But this follows easily from zig-zag property, namely $f(s_2) \xrightarrow{a} f(s_1')$ implies that exists s_2' in $T_{\doteq}$ such that $s_2 \xrightarrow{a} s_2'$ and $f(s_2') = f(s_1')$.

$\Rightarrow$: Assumed an equivalence $\sim$ to be a bisimulation, we construct an object U being the quotient of $T_{\doteq} = (S, r, tran)$ by $\sim$. We define U as $(S/_\sim, [r]_\sim, tran')$, where $[s_1]_\sim \xrightarrow{a} [s_2]_\sim$ in U iff $s_1 \xrightarrow{a} s_2' \sim s_2$ in T for some s_2'. This transition relation is well defined as $\sim$ is a bisimulation. The canonical quotient projection $\pi : T_{\doteq} \to U$, $\pi(s) = [s]_\sim$, is a morphism in TS_L, moreover $\sim\ = \ker(|\pi|)$. It is easy to verify that π satisfies the zig-zag condition. □

Due to this lemma, $\doteq_T$ is the greatest *zig-zag congruence*, that is the greatest kernel of a zig-zag morphism. One can check that strong bisimilarity $\sim_{Bran_L}$ is factorized by $\doteq$, i.e.

$$T_1 \sim_{Bran_L} T_2 \ \text{ iff } \ T_1/_{\doteq} \cong T_2/_{\doteq}. \tag{3}$$

Equivalently spelled out, strong bisimilarity $\sim_{Bran_L}$ coincides with the behavioural equivalence $\equiv_{\doteq}$ in the category of transition systems.

In Section 4 we redo this scheme in the framework of an arbitrary concrete category (zig-zag congruences are replaced by kernels of open maps), and find some conditions which are necessary and sufficient for the factorization analogous to (3) to hold. Moreover, (3) is an immediate corollary from Theorem 4.4 (see Corollary 4.7), which is applicable due to the fact that transition systems have the greatest zig-zag congruences.

While reading Section 4 it could be helpful to keep in mind algebraic observational equivalence formulated abstractly in Section 1, where indistinguishability relation was defined as the *greatest* observational congruence on the *smallest* observational subobject. In the following section we move this subobject/quotient duality to the framework of open maps.

3 Open congruences and subobjects

Let $\mathbb{M}$ be an arbitrary category of models of computation with some fixed subcategory $\mathbb{P}$ of observation objects. We assume throughout the rest of this paper that:

- category $\mathbb{M}$ has pullbacks,
- $\mathbb{M}$ is equipped with a faithful concretization functor $|\_| : \mathbb{M} \to Set^S$, for some set S,
- the concrete category $(\mathbb{M}, |\_|)$ has full subobjects (see Section 1).
- $(\mathbb{M}, |\_|)$ has surjective quotients.

Recall that existence of pullbacks in $\mathbb{M}$ implies that $\mathbb{P}$-bisimilarity is an equivalence, since pullbacks of open maps are open [JNW93]. We state here some general properties of open maps, useful in the following sections.

Proposition 3.1 *If $(f; g)$ is $\mathbb{P}$-open and g is mono, then f is also $\mathbb{P}$-open.*

Proof: To show that $f : A \to B$ is $\mathbb{P}$-open, we construct for any commuting square

$$\begin{tikzcd} O_1 \arrow[r, "p"] \arrow[d, "m"'] & A \arrow[d, "f"] \\ O_2 \arrow[r, "q"'] & B \end{tikzcd} \tag{4}$$

(with $m : O_1 \to O_2$ in $\mathbb{P}$) a diagonal morphism $r : O_2 \to A$ making both triangles commute. A good candidate is a diagonal morphism r in the commuting diagram:

$$\begin{tikzcd} O_1 \arrow[r, "p"] \arrow[d, "m"'] & A \arrow[d, "f;g"] \\ O_2 \arrow[r, "q;g"'] \arrow[ur, dotted, "r"] & B \end{tikzcd}$$

As $(f; g) : A \to C$ is open such a morphism exists and makes: $m; r = p$ and $r; f; g = q; g$; from g being mono it follows now that $r; f = q$ and so indeed f is open. $\square$

The following definition will be useful in this section as well as in the sequel:

Definition 3.2 *We say that $\mathbb{M}$ is* initialized *(w.r.t. $\mathbb{P}$) if for any $\mathbb{P}$-open morphism $h : A \to B$, any object O' in $\mathbb{P}$ and a computation $p' : O' \to B$ there exists an object O in $\mathbb{P}$ together with computation $p : O \to A$ and a morphism $m : O \to O'$ in $\mathbb{P}$ such that diagram (1) commutes.*

For instance, $\mathbb{M}$ is initialized w.r.t. $\mathbb{P}$ when $\mathbb{M}$ and $\mathbb{P}$ have a common initial object. In such a case, the unique morphism from the initial object can be thought of as the empty (initial) computation. In the proof above we rely on the fact that whenever $\mathbb{M}$ is *initialized* then for any open morphism $f : A \to B$ and a computation $q : O \to B$ there exists a computation $p : O \to A$ such that $p; f = q$.

Proposition 3.3 *Assume that $\mathbb{M}$ is initialized. If g and $(g; f)$ are $\mathbb{P}$-open, then f is $\mathbb{P}$-open as well.*

Proof: We are going to construct a diagonal morphism for any commuting diagram (4). As g is open and $\mathbb{M}$ is initialized, there exists a morphism $s_1 : O_1 \to C$ (let C be the domain of g)

such that $s_1; g = p$. Now consider a commuting square:

$$\begin{tikzcd}
O_1 \arrow[r, "s_1"] \arrow[dd, "m"'] & C \arrow[d, "g"] \\
 & A \arrow[d, "f"] \\
O_2 \arrow[r, "q"'] \arrow[uur, dotted, "s_2"] \arrow[ur, dashed, "r"'] & B
\end{tikzcd}$$

and take any diagonal morphism $s_2 : O_2 \to C$ such that $m; s_2 = s_1$ and $s_2; g; f = q$. We will show that $r = (s_2; g)$ is a diagonal morphism of diagram (4). We have $r; f = s_2; g; f = q$. Moreover, $m; r = m; s_2; g = s_1; g = p$. □

Recall from Section 1 the universal properties of quotients and subobjects that determine uniquely, for certain morphisms g, morphisms $\bar{g}$ or $\hat{g}$.

Proposition 3.4 *For any $\mathbb{P}$-open morphism $f : A \to B$, the quotient projection $\pi_{A/\ker(|f|)}$ is open. Moreover, when $\mathbb{M}$ is initialized, then for any $\mathbb{P}$-open morphism $g : A \to C$ meeting $\ker(|f|) \subseteq \ker(|g|)$, the unique morphism $\bar{g}$ satisfying $g = \pi_{A/\ker(|f|)}; \bar{g}$ is also $\mathbb{P}$-open.*

Proof: The quotient projection $\pi_{A/\ker(|f|)}$ is open by Proposition 4 as $|\bar{f}|$ is mono. The openness of $\bar{g}$ follows now from Proposition 3.3. □

Let $\mathbb{COMP}_A \subseteq |A|$ stand for the union of images of all computations in A:

$$\mathbb{COMP}_A := \bigcup_{O \xrightarrow{p} A,\ O \in |\mathbb{P}|} |p|(|O|). \tag{5}$$

Proposition 3.5 *For any $A \in |\mathbb{M}|$ and any full subobject $\iota_{B \hookrightarrow A} : B \to A$ including $\mathbb{COMP}_A$ (i.e., $\mathbb{COMP}_A \subseteq |B|$), the inclusion $\iota_{B \hookrightarrow A} : B \to A$ is $\mathbb{P}$-open. Moreover, for any open $g : C \to A$ such that $|g|(|C|) \subseteq |B|$, the unique morphism $\hat{g}$ satisfying $\hat{g}; \iota_{B \hookrightarrow A} = g$ is also $\mathbb{P}$-open.*

Proof: For any computation $q : O \to A$ (O in $\mathbb{P}$), $|q|(|O|) \subseteq |B|$, so there exists a computation $\hat{q} : O \to B$ such that $\hat{q}; \iota_{B \hookrightarrow A} = q$, as B is a full subobject. Now for any commuting diagram $p; \iota_{B \hookrightarrow A} = m; q$, where $p : U \to B$ (U in $\mathbb{P}$), $m : U \to O$ in $\mathbb{P}$, we will show that $\hat{q}$ is a diagonal morphism. We already have $\hat{q}; \iota_{B \hookrightarrow A} = q$; moreover $m; \hat{q}; \iota_{B \hookrightarrow A} = m; q = p; \iota_{B \hookrightarrow A}$, so (as $\iota_{B \hookrightarrow A}$ is mono) we obtain $m; \hat{q} = p$.

Openness of $\hat{g}$ follows immediately by Proposition 3.1. □

By the above two propositions, quotients by kernels of open maps as well as subobjects including $\mathbb{COMP}_A$ are also quotients and subobjects in the subcategory of open maps. These quotients and subobjects will play an important role in the following sections, hence for convenience we introduce the following naming convention:

open congruence on A:	a kernel of an open map from A,
open quotient:	a quotient by an open congruence,
open subobject of A:	a full subobject including $\mathbb{COMP}_A$.

Notice that by Proposition 3.4 we know that a congruence is open if and only if it has an open quotient projection. Moreover, when $\mathbb{M}$ is initialized, which is usually the case, we easily notice that a (full) subobject is open if and only if its inclusion is open. This leads to the following conclusion (for initialized $\mathbb{M}$):

open congruence	=	a congruence with an open projection,
open subobject	=	a full subobject with open inclusion.

Note the analogy to the observational congruences and subobjects in Section 1.

In the following sections we only use quotients by open congruences. Hence, instead of the fourth assumption in the beginning of this section, it is sufficient to assume that

- there exist in $\mathbb{M}$ quotients by all open congruences and moreover the quotient projections are surjective.

This will be crucial in Section 6.

4 Factorization by the greatest open congruences

In this section we address the following problem: for an arbitrary category $\mathbb{M}$, under the same assumptions as in the previous section, and for its arbitrary subcategory $\mathbb{P}$,

> *Find an indistinguishability structure (cf. Section 1), i.e., a family of partial congruences $\doteq\, = \{\doteq_A\}_{A\in|\mathbb{M}|}$, inducing the same abstract equivalence as $\mathbb{P}$-open morphisms, that is, factorizing $\mathbb{P}$-bisimilarity. That is, we want $\equiv_{\doteq} = \sim_{\mathbb{P}}$, or equivalently*
>
> $$A/_{\doteq}\cong B/_{\doteq} \iff A\sim_{\mathbb{P}}B, \quad \text{for all } A,B\in|\mathbb{M}|.$$

In the following we prove that under Assumption 4.3 the family of the greatest open congruences on the smallest open subobjects is the solution to this problem (Theorem 4.4). The domain subobjects will be defined as the full subobjects $\langle\mathbb{COMP}_A\rangle_A$ generated by the set-theoretic sum of images of all computations. We denote the subobject $\langle\mathbb{COMP}_A\rangle_A$ in short by $\widetilde{A}$, as it will be often used in the sequel.

Why do we choose the smallest open subobjects and the greatest open congruences? Restricting to open maps is suggested by Propositions 3.4 and 3.5, stating that the subcategory of open maps inherits all the relevant algebraic structure of $\mathbb{M}$, i.e., open subobjects and surjective quotients. The choice of *the smallest* open subobjects follows the intuition that a domain of an indistinguishability relation (i.e., the observable part of an object) should be equal to its part "visible" via computations. An argument for considering only open quotients is our aim to have $A/\doteq$ bisimilar to A. Another justification for *the smallest* subobjects and *the greatest* quotients is given by the observation that the equivalence induced by any of indistinguishability structures under consideration is finer than bisimilarity:

Lemma 4.1 *Let $\approx$ be an indistinguishability structure in $\mathbb{M}$ consisting, for each object A, of an open congruence $\approx_A$ on an open subobject of A. Then $\equiv_\approx \subseteq \sim_{\mathbb{P}}$.*

Proof: Assume that $A \equiv_\approx B$, that is $A/_\approx$ and $B/_\approx$ are isomorphic. Then we obtain a span of $\mathbb{P}$-open morphisms $f' = f;\iota_{A_\approx\hookrightarrow A}$ and $g' = g;\iota_{B_\approx\hookrightarrow B}$ from a pullback of the pair of quotient projections:

$$
\begin{tikzcd}
 & & C \arrow[dl, "f"'] \arrow[dr, "g"] & & \\
A & A_{\approx} \arrow[l, hook'] \arrow[dr, "\pi"'] & & B_{\approx} \arrow[r, hook] \arrow[dl, "\pi"] & B \\
 & & A/_{\approx} \cong B/_{\approx} & &
\end{tikzcd}
$$

Both f', g' are $\mathbb{P}$-open since f and g are open (because pullbacks of open morphisms are open – proof of this fact can be found in [JNW93]) and the subobject inclusions $\iota_{A_{\approx} \hookrightarrow A}$ and $\iota_{B_{\approx} \hookrightarrow B}$ are open maps (by Proposition 3.5). $\square$

Thus, choosing the smallest open subobjects and the greatest open congruences we aim, intuitively, at staying as close as possible to the bisimilarity.

In addition, note that whenever $\equiv_{\approx}$ and $\sim_{\mathbb{P}}$ coincide, the indistinguishability structure $\approx$ is especially well-behaved:

Proposition 4.2 *If $\equiv_{\approx} = \sim_{\mathbb{P}}$, then all quotients $A/\!\approx$ are* weakly fully abstract*, i.e.,*

$$(A/\!\approx)/\!\approx \;\cong A/\!\approx.$$

Proof: Let A be an arbitrary object of $\mathbb{M}$. Since the quotient projection $\pi_{A_{\approx}/\approx_A}$ as well as the subobject inclusion $\iota_{A_{\approx} \hookrightarrow A}$ are $\mathbb{P}$-open, they form a span of open maps relating $A/\!\approx$ and A, i.e., we have $A/\!\approx\, \sim_{\mathbb{P}} A$. This implies immediately $A/\!\approx\, \equiv_{\approx} A$, i.e., $(A/\!\approx)/\!\approx\, \cong A/\!\approx$. $\square$

Now we are going to introduce formally the indistinguishability structure $\doteqdot$ to factorize $\mathbb{P}$-bisimilarity. By $\mathbb{CONGR}_A$ we denote, for any object A, the set of all open congruences on A:

$$\mathbb{CONGR}_A := \{\ker(|f|) : A \xrightarrow{f} B \text{ is } \mathbb{P}\text{-open}\}.$$

We need in the sequel the following assumption, justified by some important examples, like transition systems (cf. Section 4.1) or presheaves (Section 7):

Assumption 4.3 *For each object A in $\mathbb{M}$, there exists the greatest (w.r.t. inclusion) open congruence on $\widetilde{A}$, i.e., the poset $(\mathbb{CONGR}_{\widetilde{A}}, \subseteq)$ has the greatest element.*

For any object A in $\mathbb{M}$, let $\doteqdot_A$ be a partial congruence on A such that its domain subobject $A_{\doteqdot}$ is $\widetilde{A}$ and $\doteqdot_A$ is the greatest congruence on $A_{\doteqdot}$. Consider then the indistinguishability structure $\doteqdot\; = \{\doteqdot_A\}_{A \in \mathbb{M}}$. Note the analogy to the indistinguishability structure for observational equivalence in Section 1.

Before stating the main result (Theorem 4.4), we need some more definitions. First, we say that *computations create subobjects* in $\mathbb{M}$ if, for any A in $\mathbb{M}$,

$$|\widetilde{A}| = \mathbb{COMP}_A$$

(each element of a domain subobject is in image of some computation). Second, we say that the concretization functor $|\_|$ *lifts isomorphisms of domain subobjects* if for any open morphism $f : \widetilde{A} \to \widetilde{B}$, $|f|$ is iso (bijection) implies f is iso. This means, roughly speaking that open maps are powerful enough to transport and reflect the whole structure of observable parts of objects. In the following theorem we formulate conditions to guarantee a factorization of $\mathbb{P}$-bisimilarity. All these requirements are verified in Sections 4.1, 6 and 7 to be satisfied by some standard frameworks.

Theorem 4.4 (Factorization) *Assume that*

- $|\_|$ *lifts isomorphisms of domain subobjects,*
- *computations create subobjects in* $\mathbb{M}$*,*
- $\mathbb{M}$ *is initialized.*

Then $\equiv_{\doteqdot} = \sim_{\mathbb{P}}$*.*

Before embarking on the proof, we formulate and prove two lemmas which will be needed afterwards. In the first lemma we exhaustively check all the four possibilities of consecutively applying one of two operations: taking a domain subobject $(\_)_{\doteqdot}$ and quotient $\_/\doteqdot$ by the partial congruence. The lemma states, roughly speaking, that all four collapse, i.e., are equivalent to a single application of one of the operations, up to isomorphism:

Lemma 4.5 *If* $\mathbb{M}$ *is initialized and computations create subobjects, then for any object* A *in* $\mathbb{M}$ *it holds:*

1. $(A_{\doteqdot})_{\doteqdot} \cong A_{\doteqdot}$,
2. $(A_{\doteqdot})/\doteqdot \cong A/\doteqdot$,
3. $(A/\doteqdot)_{\doteqdot} \cong A/\doteqdot$,
4. $(A/\doteqdot)/\doteqdot \cong A/\doteqdot$.

Moreover, isomorphisms in 1 and 3 are exact, i.e., their concretizations are identities.

Proof: First, observe that it is sufficient to show 1 and 3; 2 is a direct conclusion of 1 and 4 follows from 3, since the surjective quotient $A/\doteqdot$ of $A_{\doteqdot}$ by the greatest open congruence can have no open congruences but identity.

For 1 it is sufficient to show that

$$\mathbb{COMP}_{A_{\doteqdot}} = |A_{\doteqdot}|. \tag{6}$$

Since the subobject inclusion $\iota_{A_{\doteqdot} \hookrightarrow A}$ is full, each computation in A factorizes through $A_{\doteqdot}$. From this observation we conclude

$$\mathbb{COMP}_{A_{\doteqdot}} = \mathbb{COMP}_A.$$

Now equality (6) follows immediately, since $\mathbb{COMP}_A = |A_{\doteqdot}|$, which holds due to the fact that computations create subobjects in $\mathbb{M}$.

For 3, observe that $A/\doteqdot$ is a surjective quotient of $A_{\doteqdot}$. Using (6), by surjectivity of the quotient projection we conclude $\mathbb{COMP}_{A/\doteqdot} = |A/\doteqdot|$, which implies 3. $\square$

Second lemma is crucial for the proof of Theorem 4.4:

Lemma 4.6 *Under the assumptions of Theorem 4.4, any* $\mathbb{P}$*-open morphism* $h : \widetilde{A} \to \widetilde{B}$ *is a (surjective) quotient projection and consequently,* $\widetilde{B}$ *is a quotient of* $\widetilde{A}$ *by* $\ker(|h|)$*.*

Proof: Consider a quotient C of $\widetilde{A}$ by $\ker(|h|)$, with the quotient projection $\pi : \widetilde{A} \to C$.

$$\begin{tikzcd} & C \arrow[d, "\bar{h}"] \\ \widetilde{A} \arrow[ur, "\pi"] \arrow[r, "h"'] & \widetilde{B} \end{tikzcd}$$

The unique morphism $\bar{h} : C \to \widetilde{B}$ making $\pi; \bar{h} = h$ is open by Proposition 3.4. We will show $\bar{h}$ to be iso. By the assumption that the concretization functor lifts isomorphisms of domain subobjects, it is enough to show that $|\bar{h}|$ is a bijection. Obviously, $|\bar{h}|$ is injective as $\ker(|\pi|) = \ker(|h|)$ and $|\pi|$ is surjective, hence it remains only to show $|\bar{h}|$ to be surjective.

We start by showing that $|h|$ is surjective. For any computation $q : O \to \widetilde{B}$, there exists a computation $p : O \to \widetilde{A}$ such that $p; h = q$, as h is open and $\mathbb{M}$ is initialized. Hence $|q| = |p; h|$ and $|q|(|O|) = |p; h|(|O|) \subseteq |h|(|\widetilde{A}|)$. As q is an arbitrary computation in $\widetilde{B}$, we conclude $\mathbb{COMP}_{\widetilde{B}} \subseteq |h|(|\widetilde{A}|)$. But $\mathbb{COMP}_{\widetilde{B}} = |\widetilde{\widetilde{B}}|$ because computations create subobjects in $\mathbb{M}$, and $|\widetilde{\widetilde{B}}| = |\widetilde{B}|$ by Lemma 4.5(1). From this we conclude $|\widetilde{B}| \subseteq |h|(|\widetilde{A}|)$, i.e., $|h|$ is surjective. This immediately implies $|\bar{h}|$ to be surjective as well, since $|\pi|; |\bar{h}| = |h|$. $\square$

Proof of Theorem 4.4: Inclusion $\equiv_{\bumpeq} \subseteq \sim_{\mathbb{P}}$ follows directly from Lemma 4.1, hence we only need to show $\sim_{\mathbb{P}} \subseteq \equiv_{\bumpeq}$. Consider two objects A, B in $\mathbb{M}$ such that $A \sim_{\mathbb{P}} B$, i.e., A and B are related by a span

$$\begin{tikzcd} & C \arrow[dl, "f"'] \arrow[dr, "g"] & \\ A & & B \end{tikzcd}$$

of open morphisms. The subobject inclusion $\iota_{C_{\bumpeq} \hookrightarrow C} : C_{\bumpeq} \to C$ is open by Proposition 3.5.

First we aim at showing that f induces an open map between domain subobjects $C_{\bumpeq} \to A_{\bumpeq}$. Since for any computation $p : O \to C_{\bumpeq}$, the morphism $(p; \iota_{C_{\bumpeq} \hookrightarrow C}; f)$ is a computation in A, we obtain trivially $|\iota_{C_{\bumpeq} \hookrightarrow C}; f|(|C_{\bumpeq}|) = |\iota_{C_{\bumpeq} \hookrightarrow C}; f|(\mathbb{COMP}_C) \subseteq \mathbb{COMP}_A$. The subobject $A_{\bumpeq}$ is full, thus there exists the unique morphism $g = \widehat{(\iota_{C_{\bumpeq} \hookrightarrow C}; f)} : C_{\bumpeq} \to A_{\bumpeq}$ making the square in the following diagram commutative (i.e., $g; \iota_{A_{\bumpeq} \hookrightarrow A} = \iota_{C_{\bumpeq} \hookrightarrow C}; f$):

$$\begin{tikzcd} & C_{\bumpeq} \arrow[r, hook] \arrow[dl, "\pi_{C/\bumpeq}"'] \arrow[d, "g"] & C \arrow[d, "f"] \\ C/\bumpeq & A_{\bumpeq} \arrow[l, "h"] \arrow[r, hook] & A \end{tikzcd}$$

By Proposition 3.5 we know that g is open (since both the subobject inclusion $\iota_{C_{\bumpeq} \hookrightarrow C} : C_{\bumpeq} \to C$ and f are open), hence by Lemma 4.6, $A_{\bumpeq}$ is a quotient of $C_{\bumpeq}$ and g is a quotient projection. Since $\bumpeq_C$ is the greatest congruence on $C_{\bumpeq}$, there is the unique morphism $h = \overline{\pi_{C/\bumpeq}} : A_{\bumpeq} \to C/\bumpeq$ satisfying $g; h = \pi_{C/\bumpeq}$, open by Proposition 3.4.

By Lemma 4.5(3), $C/\bumpeq$ is necessarily its own domain subobject $(C/\bumpeq)_{\bumpeq}$ (up to exact isomorphism), hence by Lemma 4.6, h is also a quotient projection and $C/\bumpeq$ is a quotient of $A_{\bumpeq}$. From $\bumpeq_C$ being the greatest among open congruences on $C_{\bumpeq}$ it follows that $C/\bumpeq$ has no open congruences but identity, i.e., $C/\doteq$ is *fully abstract* (since otherwise we would construct an open congruence on $C_{\doteq}$ strictly containing $\doteq_C$). But this implies that $\ker(|h|)$ is a maximal open congruence on $A_{\doteq}$, hence the greatest one. This gives $A_{\doteq}/\doteq_A \cong C_{\doteq}/\doteq_C$.

Analogously we obtain the quotient projection $h' : B_{\doteq} \to B_{\doteq}/\doteq_B \cong C_{\doteq}/\doteq_C$; thus we conclude that $A_{\doteq}/\doteq_A \cong B_{\doteq}/\doteq_B$, which completes the proof. □

4.1 Strong bisimilarity

We apply here Theorem 4.4 to strong bisimilarity between transition systems, i.e., $Bran_L$-bisimilarity in category TS_L. First, we should verify that this category satisfies all assumptions needed there. Category TS_L has pullbacks [NW95]. We will describe full subobjects and quotients, omitting easy verification that they satisfy the appropriate requirements. For a transition system $T = (S, i, t)$, and a subset $X \subseteq S$ of states, we define $\langle X \rangle_T$ as $(X \cup \{i\}, i, t')$ with transition relation t' obtained by restriction of t to $(X \cup \{i\}) \times (X \cup \{i\})$. For a morphism $f : T \to U$, we define the quotient of T by $\sim := \ker(|f|)$ as $(T/\sim, [i]_\sim, t')$, where a transition $[s]_\sim \xrightarrow{a} [r]_\sim$ is in t' if there are some states s', r', such that $s \sim s'$, $r \sim r'$, $s' \xrightarrow{a} r'$ in T. The canonical transformation $s \mapsto [s]_\sim$ is a (surjective) quotient projection.

For any transition system T, $\mathbb{COMP}_T$ is the set of its reachable states, hence trivially computations create subobjects in TS_L. Moreover, TS_L is initialized, as it has an initial object $(\{i\}, i, \emptyset)$, which is also initial in $Bran_L$. To prove that the forgetful functor $|\_|$ restricted to zig-zag morphisms between reachable transition systems lifts isomorphisms, it suffices to show that f^{-1} is a morphism of transition systems, which follows easily from the zig-zag property of f. Moreover, Lemma 2.1 from Section 2 ensures the existence of the greatest open congruence on a reachable subobject of a transition system. Having verified all the requirements, we conclude (as announced earlier):

Corollary 4.7 *The strong bisimilarity is factorized by the greatest bisimulations.*

5 Maximal open congruences

A careful examination of the proof of Theorem 4.4 leads to an observation that the requirement of existence of the greatest open congruences is slightly too strong – a weaker, but still sufficient assumption is formulated in Theorem 5.1 below. It turns out to be also necessary, hence we obtain a characterization of open-map bisimilarities that are factorizable by open congruences. This result is applied in the following section to event structures, failing to have the greatest open congruences.

Let $\approx$ denote an arbitrary indistinguishability structure consisting exclusively of partial open congruences with $A_\approx = \widetilde{A}$, for all objects A. The consideration below are done under the assumption that the posets $\mathbb{CONGR}_{\widetilde{A}}$ are *bounded*, that is every increasing chain in $\mathbb{CONGR}_{\widetilde{A}}$ has an upper bound (we will say in short that *congruences are bounded* in $\mathbb{M}$). Note that this guarantees that for each open congruence, there exists a maximal open congruence including it. We will say that $\mathbb{M}$ *is normalized by* $\approx$ if quotients of $A_\approx$ by all maximal open congruences are isomorphic to $A/_\approx$ (for all A in $\mathbb{M}$). Equivalently, this can be divided into two separate conditions:

- quotients of A by all maximal congruences are isomorphic,
- $\approx_A$ is a maximal congruence.

The first condition, intuitively, amounts to saying that there is essentially only one way to behaviourally quotient any object.

Theorem 5.1 *Assume that congruences are bounded in $\mathbb{M}$ and moreover, similarly as in Theorem 4.4, that $|\_|$ lifts isomorphisms of domain subobjects, computations create subobjects in $\mathbb{M}$ and $\mathbb{M}$ is initialized. Then the following conditions are equivalent:*

1. $\equiv_{\approx} = \sim_{\mathbb{P}}$,
2. $\mathbb{M}$ *is normalized by* $\approx$.

Proof: $1 \Leftarrow 2$: A slight adaptation of the proof of Theorem 4.4. Similarly as there, we construct the quotient projection $g : C_{\approx} \to A_{\approx}$ by application of Lemma 4.6. $C/{\approx}$ need not to be a quotient of $A_{\approx}$ – instead, take a maximal open congruence $\sim_{\max}$ on $C_{\approx}$ including $\ker(|g|)$.

$$\begin{tikzcd}
 & C_{\approx} \arrow[dl, "\pi_{C/\sim_{\max}}"'] \arrow[d, "g"] \\
C/{\sim_{\max}} & A_{\doteq} \arrow[l, "h"]
\end{tikzcd}$$

By an argument similar to that in Theorem 4.4 we will construct an open quotient projection $h : A_{\approx} \to C/{\sim_{\max}}$. Since $\sim_{\max}$ is maximal, $\ker(|h|)$ is maximal too. Applying twice the fact that $\mathbb{M}$ is normalized by $\approx$, we obtain $C/{\approx} \cong C/{\sim_{\max}} \cong A/\ker(|h|) \cong A/{\approx}$, which, after the analogous argument for B, implies $A/{\approx} \cong B/{\approx}$.

$1 \Rightarrow 2$: Let $A_{\max}$ be a quotient of $A_{\approx}$ by any of maximal open congruences. Since $A \sim_{\mathbb{P}} A_{\max}$ (as the quotient projection $\pi : A \to A_{\max}$ is open by Proposition 3.4), we conclude

$$A_{\max}/_{\approx} \cong A/_{\approx}. \tag{7}$$

One can easily check that $(A_{\max})_{\approx} = A_{\max}$, analogously as in Lemma 4.5(3) (in general, each object being an open quotient of a domain subobject is a domain subobject itself). From maximality of the kernel of π (in $\mathbb{CONGR}_{A_{\approx}}$) we conclude that $\approx_{A_{max}}$ is identity on $A_{\max}$, hence $A_{\max} \cong A_{\max}/_{\approx}$; now by (7) we get $A_{\max} \cong A/_{\approx}$, which completes the proof. $\square$

When all $\approx_A$ are maximal, we can strengthen Proposition 4.2:

Proposition 5.2 *Assume that computations create subobjects. If $\approx$ consists exclusively of maximal open congruences and $\equiv_{\approx} = \sim_{\mathbb{P}}$, then all quotients $A/{\approx}$ are* fully abstract, *i.e., $\approx_{A/\approx}$ is an identity, for every object A of $\mathbb{M}$.*

Proof: Let A be an arbitrary object of $\mathbb{M}$. First, observe that $\approx_{A/\approx}$ is total, and moreover $\mathbb{COMP}_{A/\approx} = |A/{\approx}|$, which is guaranteed by the assumption that computations create subobjects and by surjectivity of the quotient projection $\pi_{A/\approx} : A_{\approx} \to A/{\approx}$.

Now, assume that $\approx_{A/\approx}$ is not identity. By composing the quotient projections for $\approx_A$ and $\approx_{A/\approx}$, we obtain an open congruence $\ker(|\pi_{A_{\approx}/\approx}; \pi_{(A/\approx)/\approx}|)$ on $A_{\approx}$ strictly coarser than $\approx_A$, which contradicts the fact that $\approx_A$ is maximal in $\mathbb{CONGR}_{A_{\approx}}$. $\square$

Consequently, by Theorem 5.1 and Proposition 5.2 we conclude that under assumptions from Theorem 5.1, $\equiv_{\approx} = \sim_{\mathbb{P}}$ implies that all quotients $A/{\approx}$ are fully abstract.

6 Shp-bisimilarity

Our aim is now to apply the results from the previous section to shp-bisimilarity in the category ES_L of event structures (over a fixed, throughout this section, labelling set L). We feel free to omit prefixes Pom_L- and shp-, obvious in the present context. As the notation $|E|$ suggests, category of event structures can be equipped with a concretization functor $|\_|$, taking an event structure E to its set $|E|$ of events, and a morphism f to the underlying function $|f|$. Since morphisms of event structures are functions, we write sometimes f instead of $|f|$, if this does not lead to confusion.

First we should prove that the category of event structures satisfies all assumptions needed in Section 5. Category ES_L has pullbacks [JNW93] and the concrete category $(ES_L, |\_|)$ has full subobjects – the set of events of the full subobject of an event structure E generated by $X \subseteq |E|$ is the (downward) closure of X under the causal dependency, while the labelling, causal dependency and consistency relation are inherited from E. However, we will not need subobjects in the sequel, since $\mathbb{COMP}_E$ is the set $|E|$ of all events of E, hence $\widetilde{E} = E$ (and obviously computations create subobjects in ES_L). The category of event structures is initialized, which follows, similarly as for transition systems, from the existence of the initial event structure, carrying empty set of events. The forgetful functor $|\_|$ lifts isomorphisms of domain subobjects: for any bijective morphism f in ES_L, its openness implies that f^{-1} is also a morphism of event structures.

Instead of showing event structures to have surjective quotients, we will only show that quotients by open congruences exist and are surjective – this is sufficient for Theorem 5.1 to hold, by the remark at the end of Section 5. Surprisingly, the following proposition holds:

Proposition 6.1 *Every open morphism of event structures is a surjective quotient projection.*

Proof: First, observe that each open morphism is surjective, due to the zig-zag condition. Let E, F be a domain and a codomain of an open morphism f. We are to show that for an arbitrary morphism $g : E \to G$ satisfying $\ker(g) \supseteq \ker(f)$, the unique function $\bar{g}$ defined by $\bar{g}(f(e)) = g(e)$:

$$\begin{tikzcd} & G \\ E \arrow[ur, "g"] \arrow[r, "f"'] & F \arrow[u, "\bar{g}"'] \end{tikzcd}$$

is a morphism of event structures. But this follows immediately from the fact that an open map restricted to a configuration is an isomorphism of pomsets, cf. [JNW93], Proposition 8. Let C' be a finite configuration in F; by the zig-zag property of f we conclude that there exists a configuration C in E with $f(C) = C'$. By the fact cited above, C and C' are pomset-isomorphic; moreover we have $g(C) = \bar{g}(C')$. All this implies that $\bar{g}(C')$ is a configuration and $\bar{g}$ restricted to C' is 1-1. $\square$

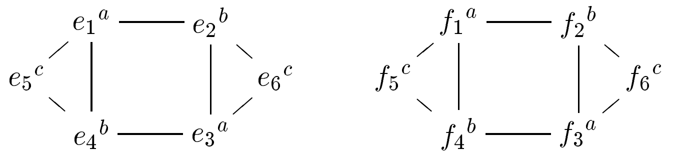
Observe that the kernel $\sim$ of any morphism *respects consistency*, i.e., it does not identify consistent events: $e_1 {\sim} e_2 \wedge e_1 \neq e_2 \Rightarrow \{e_1, e_2\} \notin Con_E$. Moreover, it is *compatible with labels*, in a sense that it contains only pairs of equally labelled events: $e_1 {\sim} e_2 \Rightarrow l_E(e_1) = l_E(e_2)$. Furthermore, each equivalence $\sim$ on the set of events of any event structure, satisfying these two conditions, is a congruence (i.e., the kernel of a morphism).

A surjective quotient by an arbitrary congruence does not necessarily exist in general – as a counterexample consider an event structure consisting of four events $\{e_1, e_2, f_1, f_2\}$ labelled identically, whose consistency relation consists only of subsets of $\{e_i, f_i\}$, for $i = 1, 2$, and causality is given by $e_1 \leq f_1, e_2 \leq f_2$. Let $\sim$ be a congruence identifying f_1 with f_2 (given, for example, as a kernel of a morphism to a tree-event pomset with discrete causality relation).

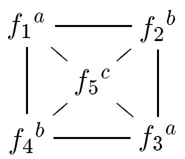
$$\begin{array}{ccccccc}
f_1 & & f_2 & \longmapsto & [f_1]_\sim & = & [f_2]_\sim \\
\uparrow\leq & & \uparrow\leq & & \uparrow\leq & & \uparrow\leq \\
e_1 & & e_2 & \longmapsto & [e_1]_\sim & & [e_2]_\sim
\end{array}$$

There is no quotient by $\sim$ in ES_L. To see this, note that in the potential quotient structure, it is not possible to have $[f_1]_\sim$ $(= [f_2]_\sim)$ causally dependent on $[e_1]_\sim$ $([e_2]_\sim)$, since the image of $\{e_2, f_2\}$ $(\{e_1, f_1\})$ should be a configuration. On the other hand, letting $[f_1]_\sim$ to be causally non-dependent on $[e_1]_\sim$ as well as on $[e_2]_\sim$ doesn't solve the problem, since a morphism $e_i \mapsto e, f_i \mapsto f$ to a two-events pomset $e \leq f$ would not factorize through the quotient projection then.

Event structures fail to have the greatest open congruences in general. A counter-example is given by an event structure with a set of events $\{e_1, \ldots, e_6, f_1, \ldots, f_6\}$ labelled by three labels a, b, c as indicated by superscripts, build up of two identical six-event components:

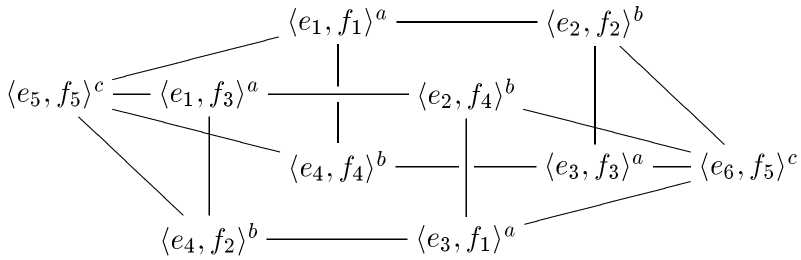


Its consistency relation contains all pairs of events connected in the picture by edges as well as all triples of events forming a triangle. The event structure has no *auto-concurrency* (no two concurrent events carry the same label) and has trivial, discrete causality relation. One can easily check that no open congruence can identify events from the same component and that there are two maximal open congruences: $\{e_1 \leftrightarrow f_1, e_2 \leftrightarrow f_2, e_3 \leftrightarrow f_3, e_4 \leftrightarrow f_4, e_5 \leftrightarrow f_5, e_6 \leftrightarrow f_6\}$ and $\{e_1 \leftrightarrow f_3, e_2 \leftrightarrow f_4, e_3 \leftrightarrow f_1, e_4 \leftrightarrow f_2, e_5 \leftrightarrow f_6, e_6 \leftrightarrow f_5\}$. In this case not everything is lost: one can check that both maximal open congruences give isomorphic quotients, which follows from auto-symmetry. Unfortunately, this observation cannot be generalized to all event structures (we correct here the result stated erroneously in Theorem 5.3 from [Las98] for all event structures) – to see this, consider a pair of event structures: E_1, one component (full substructure) of the above structure consisting of only six events $\{e_1, \ldots, e_6\}$, and E_2, the following event structure:



(8)

obtained by identifying events f_5 and f_6 in the second component above, i.e., with consistency relation containing all pairs of events connected by an edge together with triples $\{f_1, f_4, f_5\}$ and $\{f_2, f_3, f_5\}$. These two structures are bisimilar, e.g. via a bisimulation containing all pairs of pomset-isomorphic configurations, but they are not related by a co-span of open quotient projections. Hence maximal open congruences of an event structure do not have to give isomorphic quotients. More explicitly, considering the following event structure E, which represents the bisimulation between E_1 and E_2 (consistency is determined, as previously, by all edges and triangles),



(9)

kernels of two projections to E_1 and E_2 are maximal open congruences giving non-isomorphic quotients E_1 and E_2. (This event structure represents the bisimulation between E_1 and E_2 in a sense that its configurations correspond bijectively to all triples (C_1, ξ, C_2) in the bisimulation; pomset isomorphisms ξ are not explicit here, but they are determined uniquely, since both event structures under consideration have no autoconcurrency.)

These considerations motivate the searching for a subcategory of event structures whose quotients by maximal open congruences are isomorphic. In the sequel we investigate the full subcategory of *simple* structures, i.e., those $E = \langle |E|, \leq_E, Con_E, l_E\rangle$ which satisfy the following two conditions:

- *coherency*: $\forall X \subseteq^{fin} |E|.(\forall \{e, e'\} \subseteq X.\{e, e'\} \in Con_E) \implies X \in Con_E$ (i.e., consistency is essentially a binary relation),
- *uniformity*: $\forall e, e', e'' \in |E|$, whenever $e\ co_E\ e'$ and $\{e, e''\} \in Con_E$, then $\{e', e''\} \in Con_E$.

While coherent structures were widely studied (see for instance [NW95]), second condition amounts to saying that concurrent events are related in the same way to other events. This requirements rules out some not well behaved structures, for example $e_1\ co_{E_1}\ e_2$ and $\{e_1, e_5\} \in Con_{E_1}$ but $\{e_2, e_5\} \notin Con_{E_1}$. By the way, notice that E_1 is coherent while E_2 is not.

Restricting to only simple structures does not guarantee the greatest open congruences, which is caused by the possibility of auto-concurrency. In the rest of this section we show that simple event structures have isomorphic quotients by maximal open congruences (Theorem 6.7). Before proving this fact, we need to know that every open congruence is included in a maximal one, which we prove for the whole category ES_L in Lemma 6.4. The crucial role in the proof of this lemma is played by another Lemma 6.2, formulated below after some preparation.

By *the concretization* of an shp-bisimulation H between two event structures E_1 and E_2 we mean a relation $|H| \subseteq |E_1| \times |E_2|$ containing all pairs of events related by any of pomset-isomorphisms from H. Formally

$$(e_1, e_2) \in |H| \Leftrightarrow_{\text{def}} \exists\ (C_1, \xi, C_2) \in H \,\textbf{.}\, \xi(e_1) = e_2.$$

It is not possible, in general, to recover a bisimulation H from its concretization, as there can be more than one bisimulation with the same concretization. By a *lifting* $\text{Lift}(\sim)$ of a relation $\sim \subseteq |E_1| \times |E_2|$ to configurations we mean a relation consisting (similarly as shp-bisimulations) of triples (C_1, ξ, C_2), where C_1 and C_2 are finite configurations of E_1 and E_2, respectively, and ξ is an isomorphism of pomsets between them, defined by

$$(C_1, \xi, C_2) \in \text{Lift}(\sim) \ \Leftrightarrow_{\text{def}}\ \xi \subseteq \sim.$$

Intuitively, Lift($\sim$) is the greatest possible candidate to have a concretization $\sim$, but Lift($\sim$) is not necessarily a bisimulation, even if $\sim$ is a concretization of some bisimulation (for a counterexample, consider lifting of a concretization of any bisimulation in E_1 (from a counterexample above), which relates configuration $\{e_1\}$ with $\{e_3\}$ and $\{e_2\}$ with itself). We call a bisimulation H *recoverable* if $H = \text{Lift}(|H|)$. Any bisimulation satisfies $H \subseteq \text{Lift}(|H|)$ and $|\text{Lift}(|H|)| = |H|$, hence Lift(_) and $|\_|$, being monotonic, form a Galois connection between concretizations of bisimulations on one side and bisimulations together with liftings of their concretizations on the other side.

Now we are ready to state some correspondence between shp-bisimulations and open congruences. Unlike in the case of transition systems, where all bisimulation equivalences are congruences and an equivalence relation is an open congruence iff it is a bisimulation:

$$\textit{open congruence} \iff \textit{bisimulation},$$

in the category of event structures we do not even have the following correspondence

$$\textit{open congruence} \iff \textit{concretization of a bisimulation} \wedge \textit{congruence}.$$

A counterexample can be given by an equivalence relation on $|E_2|$ in (8) containing pairs (f_1, f_3) and (f_2, f_4), which is a concretization of a bisimulation and a congruence, but not an open congruence. Open congruences on an event structure correspond to recoverable bisimulations:

$$\textit{open congruence} \iff \textit{concretization of a recoverable bisimulation} \wedge \textit{congruence}.$$

Since a congruence is any equivalence which is compatible with labels and respects consistency, this can be spelled out equivalently as follows:

Lemma 6.2 *Let $\sim \subseteq |E| \times |E|$ be an equivalence on the set of events of an event structure E; $\sim$ is an open congruence on E if and only if*

(i) $\sim$ is a concretization of some recoverable bisimulation, and

(ii) $\sim$ respects consistency*: $e_1 \sim e_2 \wedge e_1 \neq e_2 \Rightarrow \{e_1, e_2\} \notin Con_E$.*

Proof: "only if": Let $\sim$ be a kernel of an open map f, i.e., $\sim \, = f^{-1} \circ f$. Applying the definition of Lift(_) to $\sim$ we get:

$$(C_1, \xi, C_2) \in \text{Lift}(\sim) \;\text{ iff }\; f(C_1) = f(C_2) \;\text{ and }\; \xi = (f\restriction_{C_2})^{-1} \circ (f\restriction_{C_1}). \tag{10}$$

Recall that f restricted to a configuration is a pomset isomorphism (which guarantees that ξ is a pomset isomorphism as well). This, together with the zig-zag property of f guarantees that Lift($\sim$) is an shp-bisimulation. Moreover, we easily conclude that $|\text{Lift}(\sim)| = \sim$, since each pair of events $e_1 \sim e_2$ identified by $\sim$ is related by some ξ from (10), for example for $C_1 = \lceil e_1 \rceil$ and $C_2 = \lceil e_2 \rceil$. Finally, kernel of any morphism respects consistency.

"if": Let $\sim$ be any binary equivalence satisfying (i) and (ii). Since $\sim$ is a concretization of a recoverable bisimulation, Lift($\sim$) is a bisimulation and $|\text{Lift}(\sim)| = \sim$. To show that $\sim$ is an open congruence, we will prove that the function $f : |E| \to |E|/\!\sim\, : e \mapsto [e]_\sim$ is an open morphism. But first, as the codomain of f we take an event structure $F = (|E|/\!\sim, \leq_F, Con_F, l_F)$ defined by:

$$\begin{aligned} \{[e_1]_\sim, \ldots, [e_k]_\sim\} \in Con_F &\Leftrightarrow_{\text{def}} \exists\, e'_1 \sim e_1, \ldots, e'_k \sim e_k . \{e'_1, \ldots, e'_k\} \in Con_E; \\ [e_1]_\sim \leq_F [e_2]_\sim &\Leftrightarrow_{\text{def}} \exists\, e'_1 \sim e_1, e'_2 \sim e_2 . e'_1 \leq_E e'_2; \\ l_F([e]_\sim) &= l_E(e). \end{aligned}$$

F is an event structure – it can be easily checked due to the following fact:

Fact 6.3 *Whenever* $e_1 \sim e_2$ *and* $e_1' \leq_E e_1$*, there exists some* e_2' *with* $e_2' \leq_E e_2$ *and* $e_1' \sim e_2'$.

Proof: Follows immediately since lifting of $\sim$ is a bisimulation and relates $\lceil e_1 \rceil$ with $\lceil e_2 \rceil$. $\square$

Now we show that f is a morphism of event structures. It is 1-1 restricted to a configuration, since $\sim$ respects consistency. To prove that the image of a configuration is a configuration we only need to show downward closure. It is sufficient to check that whenever $[e_1]_\sim \leq_F [e_2]_\sim$, then for any e_2' such that $e_2' \sim e_2$, there exists some e_1' such that $e_1' \leq_E e_2'$ and $e_1' \sim e_1$. But this follows from Fact 6.3.

Finally, we show that f is open. The very definition of the quotient structure, together with Fact 6.3, implies that any configuration C of E is pomset-isomorphic to its image $f(C)$. From this we conclude that f reflects consistency.

For the zig-zag condition of f, choose an arbitrary finite configuration C in E such that $f(C) \xrightarrow{a} C'$ in F. We should find some C'' in E with $C \xrightarrow{a} C''$ and $f(C'') = C'$. Since Lift($\sim$) is a recoverable bisimulation, (10) also holds now. By the definition of Con_F, there is some $D' \subseteq |E|$ such that $D' \in Con_E$ and $f(D') = C'$. By the definition of $\leq_F$, D' is downward closed, i.e., D' is a configuration of E. Moreover, D' has a subconfiguration D isomorphic to C, such that $f(D) = f(C)$. Now, since Lift($\sim$) is a bisimulation, by (10) we conclude existence of an appropriate C'', which completes the proof. $\square$

Lemma 6.4 *In the category of event structures, congruences are bounded.*

Proof: Consider an increasing chain of open congruences of an event structure, $\sim_1 \subseteq \sim_2 \subseteq \ldots$. By Lemma 6.2 the corresponding recoverable shp-bisimulations $H_1 := \text{Lift}(\sim_1), H_2 := \text{Lift}(\sim_2), \ldots$, satisfy $|H_i| = \sim_i$, for $i \geq 1$.

Let $H := \cup_{i \geq 1} H_i$ be the sum of all these bisimulations. Obviously H, union of a family of bisimulations, is itself a bisimulation. We will show that $\sim := |H|$ is an open congruence, being a bound of $\sim_1 \subseteq \sim_2 \subseteq \ldots$. To use Lemma 6.2, we are going to show first that H, union of a chain of recoverable bisimulations, is itself recoverable, i.e.,

$$H = \text{Lift}(\sim). \tag{11}$$

Since $|\_|$ and Lift($\_$) form a Galois connection, we have

$$\sim \, = |H| = |\cup_{i \geq 1} H_i| = \cup_{i \geq 1} |H_i| = \cup_{i \geq 1} \sim_i, \tag{12}$$

Assume (11) does not hold, i.e., there exists some triple (C, ξ, C') in Lift($\sim$) $\setminus H$. By the very definition of Lift($\_$) and by (12) we conclude that there exists some j with $(C, \xi, C') \in \text{Lift}(\sim_j)$; moreover, $(C, \xi, C') \notin H_j$, since $H_j \subseteq H$, from which we conclude that $\text{Lift}(\sim_j) \setminus H_j$ is nonempty. This obviously contradicts H_j being recoverable, and thus proves (11).

From (12) it follows also that $\sim$ respects consistency, hence satisfies both conditions in in Lemma 6.2, i.e., is an open congruence, which completes the proof.

Note that $\sim$ is in fact the least upper bound of the chain $\sim_1 \subseteq \sim_2 \subseteq \ldots$; thus we have actually shown that open congruences are closed under unions of ω-chains. $\square$

Having checked congruences to be bounded, we prove one more lemma, assuring that factorizability we are aiming at (see Corollary 6.8) takes place in the subcategory of simple structures:

Lemma 6.5 *Whenever E is simple, $E/\sim$ is also simple, for any open congruence $\sim$ on E. (Or equivalently: for an open map $f : E \to F$, whenever E is simple, F is simple too.)*

Proof: In the sequel we will need a lemma, following directly from the fact that $\sim$ is a concretization of some bisimulation:

> **Fact 6.6** *Whenever $e_1 \sim e_2$ and $\{e_1, e_1'\} \in Con_E$, there exists some e_2' with $\{e_2, e_2'\} \in Con_E$ and $e_1' \sim e_2'$.*

Coherency: Let $C_2 = \{[e_1]_\sim, \ldots, [e_k]_\sim\} \subseteq^{fin} |E/\sim|$ be any finite subsets of events of $E/_\sim$ with the property that each pair $[e_i]_\sim, [e_j]_\sim \in C_2$ is consistent. It follows immediately from openness of the quotient projection that for each such pair there exists a consistent pair e_i', e_j' in E with $e_i' \sim e_i$ and $e_j' \sim e_j$. Now, consecutive applications of Fact 6.6 together with uniformity of E guarantee the existence of a set $C_1 = \{e_1', \ldots, e_k'\}$ of events of E with $e_i' \sim e_i$, such that each pair of its elements is consistent. Since E is coherent, we conclude that C_1 is consistent, thus C_2, its image, is consistent too.

Uniformity: Assume $[e_1]_\sim \; co_{E/\sim} \; [e_2]_\sim$ and $\{[e_1]_\sim, [e_3]_\sim\} \in Cons_{E/\sim}$. By openness of the quotient projection and by Fact 6.6 we conclude that there exist $e_1' \sim e_1, e_2' \sim e_2, e_3' \sim e_3$ in E satisfying: $e_1' \; co \; e_2'$ and $\{e_1', e_3'\} \in Cons_E$. Since E is uniform we have $\{e_2', e_3'\} \in Cons_E$, which implies $\{[e_2]_\sim, [e_3]_\sim\} \in Cons_{E/\sim}$. □

We are able now to state the main result of this section: all simple quotients of an event structure by maximal open congruences are isomorphic.

Theorem 6.7 *Let $\sim$, $\sim'$ be two maximal open congruences of an event structure E, such that both quotients $E/\sim$, $E/\sim'$ are simple. Then the quotients are isomorphic.*

Proof: The proof will be based on an observation that both open quotient projections are split epi (i.e., have one-side inverse). (In fact all open maps with a simple codomain are split epi, since they are necessarily quotient projections.)

Let π, π' denote the quotient projections by $\sim$ and $\sim'$, respectively. We will construct an injective morphism $\iota : E/\sim \to E$ satisfying $\iota; \pi = \mathrm{id}_{E/\sim}$. Even more, we will construct a full subobject (substructure) F of E isomorphic to $E/\sim$. This will be done by consecutive applications of the fact that π, being open, reflects transitions (zig-zag property). Let F_0 be the empty event structure, and let $\iota_0 : \emptyset \to F_0$ be the empty function. We define inductively the family $F_1, F_2, \ldots$ of full substructures of E, having the sets of events determined by the following rule:

$$\begin{array}{c} |F_{k+1}| := |F_k| \cup \iota_{k+1}(S_k), \text{ where} \\ S_k := \{e \in |E/\sim| : e \notin \pi(|F_k|) \wedge \lceil e \rceil \setminus \{e\} \subseteq \pi(|F_k|)\} \end{array} \tag{13}$$

(intuitively, S_k contains k-th "level" of $|E/\sim|$, i.e., all events of "height" k) where

$$\iota_{k+1} : \pi(|F_k|) \cup S_k \to |E|$$

is a function extending ι_k and returning, for each $e \in S_k \subseteq |E/\sim|$, some event $f \in |E|$ satisfying

$$\pi(\lceil f \rceil) = \lceil e \rceil \text{ and } \lceil f \rceil \setminus \{f\} \subseteq |F_k|, \tag{14}$$

and moreover ι_{k+1} satisfies the following condition, crucial for correctness of induction:

$$\iota_{k+1}(C) \text{ is a configuration, for each configuration } C \text{ in } \pi(|F_k|) \cup S_k. \tag{15}$$

Observe that some value $\iota_{k+1}(e)$ satisfying (14), for each $e \in S_k$, exists by the zig-zag property of π, although it is not determined uniquely, and in general Axiom of Choice is needed here. Moreover, requirement (15) can be fulfilled since $E/\!\sim$ is simple – this guarantees that (binary) consistency relation in S_k (coinciding in S_k with concurrency relation, since no two events from the same "level" can be related by causality) is an equivalence (it is transitive due to uniformity of $E/\!\sim$), hence value of ι_{k+1} can be chosen independently in each equivalence class.

In the sequel we will impose some additional requirements on ι_{k+1} – they are postponed, by now, for simplicity reason.

Recall that π restricted to any configuration $\lceil e \rceil$, for $e \in |E|$, is an isomorphism of pomsets (cf. [JNW93]). Hence the full substructure F, of E, being the union of all F_k, i.e., determined by $|F| = \cup_{k \geq 0} |F_k|$, is isomorphic to $E/\!\sim$. Indeed, taking $\iota = \cup_{k\geq 0} \iota_k : E/\!\sim \,\to E$, we obtain $\widehat{\iota} : E/\!\sim \,\to F$ as the unique morphism making $\widehat{\iota}; \iota_{F \hookrightarrow E} = \iota$. $\widehat{\iota}$ is an isomorphism with inverse $\iota_{F\hookrightarrow E}; \pi$.

$$\begin{tikzcd}
 & F \arrow[d, hook] \arrow[dr, "\pi'\restriction_F"] & \\
 & E \arrow[dl, bend left, "\pi"] \arrow[dr, "\pi'"'] & \pi'(F) \arrow[d, hook] \\
E/\!\sim \arrow[uur, bend left, "\widehat{\iota}"] \arrow[ur, bend left, "\iota"] & & E/\!\sim'
\end{tikzcd} \tag{16}$$

We will show now that π' restricted to $|F|$, i.e., considered as a morphism $F \to \pi'(F)$ and denoted by $\pi' \restriction_F$, is an open map; note that its codomain is a full substructure of $E/\!\sim'$. Obviously it reflects concurrency, since π' does so on whole E. To show the zig-zag property, consider any configuration C in F with $\pi'(C) \xrightarrow{a} C'$ in $\pi'(F)$ – we search for a configuration C'' in F satisfying

$$C \xrightarrow{a} C'' \text{ and } \pi'(C'') = C'. \tag{17}$$

In the following we rely on the fact that the projections, being open, are pomset isomorphisms on configurations. Moreover, we implicitly exploit injectivity of ι.

Consider the family $\mathcal{C}$ of all those configurations D in E with $\pi'(D) = C'$, necessarily nonempty. We know that $C' \subseteq \pi'(F)$, hence some $D \in \mathcal{C}$ must be included in F, $D \subseteq F$. By now, none of the assumptions guarantee that the unique subconfiguration of any such D, say D', satisfying $\pi'(D') = \pi'(C)$, is equal to C. Thus we strengthen requirements on construction of ι_{k+1} in the inductive definition above. We require additionally that ι_{k+1} is chosen to satisfy the following condition:

> For any two isomorphic configurations $B_1, B_2 \subseteq \pi(F_k)$ in $E/\!\sim$ with $\pi'(\iota_k(B_1)) = \pi'(\iota_k(B_2))$, whenever $B_1 \xrightarrow{a} B_1'$ and $B_1' \not\subseteq \pi(F_k)$, there exists some B_2' with $B_2 \xrightarrow{a} B_2'$, $B_2' \not\subseteq \pi(F_k)$ and $\pi'(\iota_{k+1}(B_1)) = \pi'(\iota_{k+1}(B_2))$.

To ensure that this can be done, note that π and π' *represent* an shp-bisimulation, say H, containing all pairs $(\pi(C), \pi'(C))$ of configurations, for any configuration C of E, linked by pomset isomorphisms induced by $\pi^{-1};\pi'$ (like an event structure in (9) together with the projections to E_1 and E_2):

$$H = \{(\pi(C), (\pi' \restriction_C) \circ (\pi \restriction_C)^{-1}, \pi'(C)) : C \text{ is a configuration in } E\}.$$

Having this, observe that $\pi'(\iota(B_1)) = \pi'(\iota(B_2))$ implies that B_1 and B_2 are related (by H) to the same configuration in $E/{\sim'}$, from which we conclude that some extensions of B_1 and B_2 (via $\_ \xrightarrow{a} \_$) are also related to the same configuration of $E/{\sim'}$, hence some (not necessarily unique) ι_{k+1} can be chosen to satisfy the condition above.

This additional requirement guarantees existence of a configuration C'' satisfying (17): put $B_1' := \pi(D)$ (for any $D \in \mathcal{C}$), $B_1 := \pi(D')$ (where the configuration $D' \subseteq D$ is such that $\pi'(D') = \pi'(C)$), $B_2 := \pi(C)$, and now $C'' := \iota_{k+1}(B_2')$.

Returning to diagram (16), by the above considerations we know that $\pi' \restriction_F$ is open, hence

$$(\widehat{\iota}; \pi' \restriction_F) : E/{\sim} \to \pi'(F)$$

is also open by Proposition 3.5. Moreover, this morphism is necessarily injective, since otherwise we would obtain an open congruence $\ker(|\pi; \widehat{\iota}; (\pi' \restriction_F)|)$ strictly coarser than $\ker(|\pi|)$, which is impossible by assumption that $\ker(|\pi|)$ is maximal. By Lemma 4.6 we conclude also that $(\widehat{\iota}; \pi' \restriction_F)$ is surjective. All this implies that it is an isomorphism between $E/{\sim}$ and $\pi'(F)$, since the functor $|\_|$ lifts isomorphisms of domain subobjects.

Repeating analogously the whole argument we construct another inverse $\iota' : E/{\sim'} \to E$ and the full subobject F' of E such that $(\widehat{\iota'}; \pi \restriction_{F'}) : E/{\sim'} \to \pi(F')$ is an isomorphism. This gives the following picture,

$$\begin{tikzcd}
E/{\sim} \arrow[r, phantom, "\cong"] & \pi'(F) \arrow[d, hook] \\
\pi(F') \arrow[u, hook] \arrow[r, phantom, "\cong"] & E/{\sim'}
\end{tikzcd}$$

with $E/{\sim}$ being isomorphic to a full subobject of $E/{\sim'}$ and vice versa. Now our aim is to show that (at least) one of subobjects inclusions is an isomorphism. For this, we need in the construction of ι' one more requirement: whenever an event e of $E/{\sim'}$ is in $\pi'(F)$, i.e., $e = \pi' \restriction_F(e')$ for some e' in F, put $\iota'(e) = e'$. This is always possible, since $(\widehat{\iota}; \pi' \restriction_F)$ is an isomorphism and that each pair of configurations from $E/{\sim}$ and $\pi'(F)$, respectively, related by this isomorphism, belongs to the bisimulation H represented by π and π' (together with the evident pomset-isomorphism). Now, we have $F \subseteq F'$, hence $\pi(F') \cong E/{\sim}$, which implies $E/{\sim} \cong E/{\sim'}$ and completes the proof. $\square$

Lemma 6.4 and Theorem 6.7 enable us to apply Theorem 5.1:

Corollary 6.8 *Shp-bisimilarity of simple event structures is factorized by any indistinguishability structure consisting of maximal open congruences.*

We also considered other possibilities for a concretization functor, for instance functor $\mathrm{Conf}(\_)$ transporting an event structure E to the set of all configurations of E, and on morphism $f : E_1 \to E_2$ giving a function transporting a configuration C to $f(C)$. For $\mathrm{Conf}(\_)$ it is no longer true that computations create subobjects, but this problem disappears when we change to a functor giving only the set of *finite* configurations. The latter one could be also used here, although it would not allow us to easily keep track of concretizations of bisimulations, appearing crucial when dealing with open congruences.

7 Presheaves

Category of presheaves, i.e., set-valued functors, gives another example of application of results from the previous sections. Presheaves were proposed as a general and uniform model for concurrency [JNW93, CW96]. For an arbitrary category $\mathbb{P}$, the category $\widehat{\mathbb{P}}$ of presheaves over $\mathbb{P}^{op}$ consists of all contravariant functors from $\mathbb{P}$ to Set with natural transformations as morphisms between them.

It seems appropriate to consider presheaves over $\mathbb{P}^{op}$, for a subcategory $\mathbb{P}$ of observation objects. For example, category $\widehat{Bran_L}$ corresponds to synchronization forests, i.e., collections of *synchronization trees* (unfoldings of transition systems) (cf. [JNW93]). A presheaf F over $\mathbb{P}^{op}$ is a particularly simple model of behaviour. Its value $F(P)$ on an object P from $\mathbb{P}$ corresponds bijectively, by Yoneda Lemma, to the set of computations $P \to F$ (we treat here $\mathbb{P}$ as a subcategory of $\widehat{\mathbb{P}}$, as it embeds fully and faithfully into $\widehat{\mathbb{P}}$ via Yoneda embedding $P \mapsto Hom_{\mathbb{P}}(\_, P)$), i.e., the set of natural transformations $Hom_{\mathbb{P}}(\_, P) \to F$. Moreover, $F(m)$, for a morphism $m : P \to Q$ in $\mathbb{P}$, describes how "longer" computations from Q restrict uniquely to "shorter" ones (via m). This demonstrates uniformity and simplicity of presheaf models, which consist (up to the natural bijection) exclusively of their own computations and computation extensions.

For any category $\mathbb{M}$ and an observation subcategory $\mathbb{P}$, we can consider a canonical functor $\mathbb{M} \to \widehat{\mathbb{P}}$, taking an object X of $\mathbb{M}$ to the representable functor $Hom_{\mathbb{M}}(\_, X) : \mathbb{P}^{op} \to Set$. For transition systems we have a pleasing situation: the subcategory of synchronization trees is equivalent via such a canonical functor to subcategory of *rooted* presheaves, i.e., those presheaves that assign a singleton set to the initial object in $Bran_L$.[2] For event structures the picture is more complicated and until recently there was no convenient characterization of those presheaves over Pom_L which correspond to event structures – only quite recently such a characterization was proposed in [Win99].

One of advantages of considering presheaves $\mathbb{P}^{op} \to Set$ instead of concrete models is that many universal constructions like limits or colimits are obtained by a point-wise construction, hence similarly as in Set. However, results obtained for presheaves, like $\widehat{Bran_L}$ or $\widehat{Pom_L}$, can not always be translated to concrete categories.

From now on we consider presheaves over $\mathbb{P}^{op}$, for an arbitrary fixed small category $\mathbb{P}$. In the following we consider the class of open maps induced in $\widehat{\mathbb{P}}$ by $\mathbb{P}$ (when the latter is identified with a subcategory of $\widehat{\mathbb{P}}$); we often omit prefix $\mathbb{P}$-, and write in short "open" instead of "$\mathbb{P}$-open". Yoneda Lemma gives a nice characterization of open morphisms in $\widehat{\mathbb{P}}$ [JM94]: a natural transformation $f : F \to G$ is open iff all naturality squares[3]

$$\begin{array}{ccc} FD & \xrightarrow{f_D} & GD \\ {\scriptstyle F\alpha}\downarrow & & \downarrow{\scriptstyle G\alpha} \\ FC & \xrightarrow{f_C} & GC \end{array}$$

are weak pullbacks in Set, for all $\alpha : C \to D$ in $\mathbb{P}$ (that is, for all $x \in FC$ and $y \in GD$ with $f_C(x) = G\alpha(y)$, there exists a not necessarily unique $z \in FD$ such that $F\alpha(z) = x$ and $f_D(z) = y$). We will rely on this characterizations in the proofs below.

At the expense of slight artificiality, we introduce a concretization functor $\|\_\|$ from $\widehat{\mathbb{P}}$ to $Set^{|\mathbb{P}|}$, category of sets indexed by $|\mathbb{P}|$. (Recall that $|\mathbb{P}|$ denotes the set of objects of $\mathbb{P}$.) This functor essentially forgets about values of presheaves on morphisms, and assigns to F the family $\|F\| = \{F(C)\}_{C \in |\mathbb{P}|}$ of sets.

[2] This notion is meaningful for any category $\mathbb{P}$, assumed that $\mathbb{P}$ has an initial object.

[3] Throughout this section, to make the notation more concise, we omit brackets and write FP for application.

Unique morphisms from the empty initial presheaf are always open, hence they should be ruled out by any appropriate concept of bisimilarity in $\widehat{\mathbb{P}}$ built up on spans of open maps. There are two ways to obtain a meaningful and nontrivial notion of bisimilarity [JNW93]. One is to restrict attention to rooted presheaves. Another way around, preferred here (as more general), is to consider only spans of *surjective* open maps in the category of all presheaves. Hence, we say that two presheaves are bisimilar if they are related by a span of surjective open maps. For synchronization trees and event structures both approaches are equivalent; they are also equivalent for the subcategory of rooted presheaves, since open maps between rooted presheaves are necessarily surjective.

We will show that every presheaf has the greatest open congruence. To this aim we first establish a correspondence between open congruences and bisimulation equivalences (Lemma 7.4 below).

Definition 7.1 *By ($|\mathbb{P}|$-sorted)* bisimulation relation *we mean any relation represented by a span of surjective open maps, where a span*

$$
\begin{tikzcd}
 & H \arrow[dl, "f"'] \arrow[dr, "g"] & \\
F & & G
\end{tikzcd}
$$

represents the relation: $\{(\|f\|(s), \|g\|(s)) : s \in \|H\|\}$.

Following the standard lines, we say that F and G are bisimilar if there exists any bisimulation relating them (or equivalently, if they are related by a span of surjective open maps).

The following two lemmas concerning colimits in $\widehat{\mathbb{P}}$ will be useful in the sequel. The first lemma is a slight generalization, in the setting of presheaves, of the fact that quotients by open congruences are open (see Proposition 3.4).

Lemma 7.2 *For any parallel pair of open maps $f, g : F \to G$ between presheaves, their coequalizer $c : G \to C$ is open. Moreover, for any other open morphism $d : G \to D$ making $f; d = g; d$, the universal map $\bar{d}$ meeting $c; \bar{d} = d$ is also open.*

Proof: In the proof we rely on the weak-pullback characterization of open maps. Moreover, we use the quotient construction of colimits in *Set*, as colimits in $\widehat{\mathbb{P}}$ are point-wise.

To prove that $c : G \to C$ is open, we should show that the right-most square in the following diagram is a weak pullback, for any $h : Y \to X$ in $\mathbb{P}$:

$$
\begin{tikzcd}
FX \arrow[r, "f_X, g_X"] \arrow[d, "Fh"'] & GX \arrow[r, "c_X"] \arrow[d, "Gh"'] & CX = GX/{\sim_X} \arrow[d, "Ch"'] \\
FY \arrow[r, "f_Y, g_Y"] & GY \arrow[r, "c_Y"] & CY = GY/{\sim_Y}
\end{tikzcd}
$$

By the construction of coequalizers in *Set*, $\sim_X$ (and similarly $\sim_Y$) is the smallest equivalence relation on GX induced by the pair of functions f_X, g_X, that is, the smallest equivalence including $\{(f_X(z), g_X(z)) : z \in FX\}$. For instance, $\sim_Y = (\sim^1_Y)^*$ is a reflexive-transitive closure of a one-step relation $\sim^1_Y$ defined by:

$$y_1 \sim^1_Y y_2 \Leftrightarrow \exists z \in FY \,.\, (f_Y(z) = y_1 \wedge g_Y(z) = y_2) \vee (f_Y(z) = y_2 \wedge g_Y(z) = y_1).$$

By the point-wise construction of colimits in $\widehat{\mathbb{P}}$ we have

$$Ch([x]_{\sim_X}) = [y]_{\sim_Y} \text{ iff } \forall x' \in [x]_{\sim_X} \,.\, Gh(x') \in [y]_{\sim_Y}.$$

Hence, in other words, we should prove the following:

$$\forall y \in GY \,.\, \forall x' \in GX \,.\, Gh(x') \in [y]_{\sim_Y} \Rightarrow \exists x \in GX \,.\, x {\sim_X} x' \wedge Gh(x) = y. \tag{18}$$

For arbitrary $y \in GY$ and $x' \in GX$, let $y' := Gh(x')$. We have $y' {\sim_Y} y$, i.e., $y'(\sim^1_Y)^* y$. Assume $y' {\sim^1_Y} y$ (the general case follows by an easy induction), that is we have $z \in FY$ with $f_Y(z) = y' \wedge g_Y(z) = y$. Since f is open there exists $z' \in FX$ satisfying $Fh(z') = z$ and $f_x(z') = x'$. Now $x := g_X(z')$ satisfies $Gh(x) = y$ by commutativity of the square $g_X; Gh = Fh; g_Y$ and moreover $x' {\sim_X} x$ due to z' as $f_X(z') = x'$ and $g_X(z') = x$. This proves (18) and consequently implies that $c : G \to C$ is open.

To show openness of the unique mediating morphism $\bar{d}$, for an arbitrary open map $d : G \to D$, consider a diagram consisting of two commuting squares:

$$\begin{array}{ccccc}
GX & \xrightarrow{c_X} & CX & \xrightarrow{\bar{d}_X} & DX \\
{\scriptstyle Gh}\downarrow & & {\scriptstyle Ch}\downarrow & & {\scriptstyle Dh}\downarrow \\
GY & \xrightarrow{c_Y} & CY & \xrightarrow{\bar{d}_Y} & DY
\end{array}$$

We need to show that the right-most square is a weak pullback assumed that the left-most square and the whole rectangle are weak pullbacks. Take any $x \in DX$ and $y \in CY$ with $Dh(x) = \bar{d}_Y(y)$. As c_Y is surjective and $d = c; \bar{d}$ is open, there exists $z \in GX$ such that $c_Y(Gh(z)) = y$ and $d_X(z) = x$. But now $z' := c_X(z) \in CX$ satisfies $\bar{d}_X(z') = x$ and $Ch(z') = c_Y(Gh(z)) = y$, hence the right-most square is a weak pullback. □

Lemma 7.3 *For an arbitrary family $\{F_i\}_{i\in\mathbb{I}}$ of presheaves in $\widehat{\mathbb{P}}$, the coproduct injections $\iota^i : F_i \to (\coprod_{i\in\mathbb{I}} F_i)$ are open, and moreover the universal mediating morphism*

$$\bar{g} : (\coprod_{i\in\mathbb{I}} F_i) \to G$$

is open, for any cocone $\{g_i : F_i \to G\}_{i\in\mathbb{I}}$ of open maps.

Proof: One can easily check that coproduct injections are open. The proof of openness of $\bar{g}$ is analogous to the second part of proof of Lemma 7.2 and amounts to showing that the right-most square in the diagram below is a weak pullback. This can be done similarly as before after observation that coproduct injections are "jointly surjective" that is for each $y \in \coprod_{i\in\mathbb{I}} F_i Y$ there exists some $k \in \mathbb{I}$ with $y \in F_k Y$:

$$\begin{array}{ccccc}
F_k X & \xrightarrow{\iota^k{}_X} & \coprod_{i\in\mathbb{I}} F_i X & \xrightarrow{\bar{g}_X} & GX \\
{\scriptstyle F_k h}\downarrow & & {\scriptstyle \coprod_{i\in\mathbb{I}} F_i h}\downarrow & & {\scriptstyle Gh}\downarrow \\
F_k Y & \xrightarrow{\iota^k{}_Y} & \coprod_{i\in\mathbb{I}} F_i Y & \xrightarrow{\bar{g}_Y} & GY
\end{array}$$

□

The above two lemmas can be generalized to all colimits: the colimit cocone of any diagram consisting of open maps is open; the universal morphism to any cocone (over the same diagram) consisting of open maps is also open.

Lemma 7.4 *Let $\sim\ \subseteq \|F\| \times \|F\|$ be a ($|\mathbb{P}|$-sorted) equivalence relation. $\sim$ is a bisimulation between F and F itself (cf. Definition 7.1) iff it is an open congruence.*

Proof: An arbitrary open congruence $\sim$ on F is a bisimulation since it is represented by a kernel pair of the projection $\pi_{F/\sim}$ that is by a pullback of $\pi_{F/\sim}$ along itself. The pullback morphisms are open (cf. [JNW93]) and obviously surjective.

For the opposite direction, having a bisimulation equivalence $\sim$, we apply Lemma 7.2 to the coequalizer $c : F \to H$ of a pair $f, g : G \to F$ of open surjective maps representing $\sim$. Since colimits are point-wise in $\widehat{\mathbb{P}}$, we conclude that $\sim\ = \ker(\|c\|)$, i.e., $\sim$ is the kernel of the coequalizer c, and $H = F/\sim$. $\square$

As a straightforward corollary from Lemma 7.3 we obtain the greatest bisimulations in $\widehat{\mathbb{P}}$:

Lemma 7.5 *There exists the greatest bisimulation between any two bisimilar presheaves.*

Proof: For any two presheaves F and G, we will construct a pair $F \xleftarrow{f} H \xrightarrow{g} G$ of open maps to represent the greatest bisimulation. Let H be a coproduct of all bisimulations between F an G, i.e., $H := \coprod_{i \in \mathbb{I}} H_i$, where $\mathbb{I}$ indexes the family $\{A \xleftarrow{f_i} H_i \xrightarrow{g_i} B\}_{i \in \mathbb{I}}$ of all spans representing bisimulations, i.e., all spans of surjective open maps relating A and B. To keep this family a small set observe that it is sufficient to take one span for each bisimulation, for example, by Lemma 7.4, pullbacks of open quotient projections. Now f and g are constructed as the universal morphism from $\coprod_{i \in \mathbb{I}} H_i$, for the cocone $\{f_i : H_i \to A\}_{i \in \mathbb{I}}$ and $\{g_i : H_i \to B\}_{i \in \mathbb{I}}$, respectively. They are open by Lemma 7.3 and surjective, due to an observation that the universal morphism from a coproduct, for a cocone of open maps, is surjective whenever all cocone morphisms are surjective. $\square$

Lemmas 7.4 and 7.5 ensure the existence of the greatest open congruences $\simeq_F$ in $\widehat{\mathbb{P}}$, which allows us to apply Theorem 4.4. But first we must verify whether all the assumptions from Section 4 are satisfied in this case. One can easily construct full subobjects and surjective quotients, hence we have the domain subobjects $\widetilde{F}$ required in Assumption 4.3. Similarly as in the case of event structures, there holds

Proposition 7.6 *For all presheaves F, $\mathbb{COMP}_F = \|F\|$ (which implies $\widetilde{F} = F$ and computations create subobjects in $\widehat{\mathbb{P}}$).*

Proof: Follows from the proof of Yoneda Lemma: the 1-1 correspondence between the set of natural transformations $Nat(Hom_{\mathbb{P}}(\_, X), F)$ and FX ensures that a computation from $Hom_{\mathbb{P}}(\_, X)$ to F can map id_X to any element of FX. $\square$

Moreover, $\|\_\|$ lifts isomorphisms of domain subobjects, but in general $\widehat{\mathbb{P}}$ is not necessarily initialized. This is caused by the empty initial presheaf, hence the solution is to restrict to only rooted presheaves:

Corollary 7.7 *When $\mathbb{P}$ has the initial object, bisimilarity in the full subcategory of rooted presheaves is factorized by the greatest bisimulations.*

Proof: Theorem 4.4 applies, since Lemmas 7.2–7.5, although formulated for the whole category $\widehat{\mathbb{P}}$, hold also for the subcategory of rooted presheaves. $\square$

Recall that Corollary 7.7 applies immediately to strong bisimilarity in the category of synchronization trees, equivalent to the category of rooted presheaves over $Bran_L$.

It is worthwhile to mention that the result can be easily adapted to any suitable subcategory S of $\widehat{\mathbb{P}}$ satisfying requirements from Section 4. In addition, we need S to be closed on those universal constructions which are involved in the proofs that is pullbacks, coproducts and coequalizers, or we need appropriate colimits in S, possibly different that in $\mathbb{P}$, to still satisfy Lemma 7.3 and 7.2 (and pullbacks of open maps to be open). For example, when $\mathbb{P}$ has the initial object ι, all considerations (and Corollary 7.7) stay valid for the full subcategory of presheaves having a nonempty value on ι.

8 Domain subobjects as colimits of computations

Similarly as in Section 4, let $\mathbb{M}$ denote an arbitrary category with a subcategory $\mathbb{P}$ of observation objects; we assume moreover $\mathbb{P}$ to be small. We investigate a different possibility to define domain subobjects than in Section 4. A domain subobject of an object A will be produced now, roughly speaking, as the colimit of all computations in A. The justification for this study is that now the analog of Lemma 4.6, crucial in Theorem 4.4, holds without assumption needed there. On the other hand, the domain subobjects constructed here are not necessarily full, hence they can not be directly used in Theorem 4.4. Fortunately, if they are full, they necessarily coincide with those considered in Section 4.

In the sequel we use the following notation. For any object A in $\mathbb{M}$ we consider the *category of $\mathbb{P}$-computations* of A, denoted below by $\int_{\mathbb{P}}^{\mathbb{M}} A$. Its objects are all pairs (P, p), where P is an object of $\mathbb{P}$ and $p : P \to A$ in $\mathbb{M}$. Its morphisms $(P, p) \to (Q, q)$ are those morphisms $m : P \to Q$ from $\mathbb{P}$ for which $p = m; q$. This category can be equivalently defined as the comma category of the inclusion $\mathbb{P} \hookrightarrow \mathbb{M}$ and the constant functor $1 \mapsto A : \mathbf{1} \to \mathbb{M}$.

Since $\mathbb{M}$ and $\mathbb{P}$ are fixed in this section, we omit sometimes sub- and superscripts and write $\int A$ for the category of computations. In the following we need the projection functor $\Delta_A : \int A \to \mathbb{P}$, taking a pair (P, p) to P and a morphism $(P, p) \xrightarrow{m} (Q, q)$ to the same morphism $P \xrightarrow{m} Q$ in $\mathbb{P}$. This functor is/induces a diagram in $\mathbb{P}$ and hence also in $\mathbb{M}$. Throughout this section we need the following assumption, slightly weaker than cocompletness of $\mathbb{M}$:

Assumption 8.1 *For any object A in $\mathbb{M}$, the diagram*

$$(\int_{\mathbb{P}}^{\mathbb{M}} A) \xrightarrow{\Delta_A} \mathbb{P} \hookrightarrow \mathbb{M}$$

has a colimit, denoted in the sequel by $\mathrm{colim}\Delta_A$.

Similarly as before, we assume that category $\mathbb{M}$ is initialized and is equipped with a concretization functor $|\_| : \mathbb{M} \to Set^S$, for some set S; moreover, we assume that the concrete category $(\mathbb{M}, |\_|)$ has full subobjects and surjective quotients and admits renaming of elements (see Section 1).

Colimit of the diagram Δ_A is intuitively a "computational image" of A, containing, roughly speaking, the whole information about A, which can be gained by observing it via computations (see Example 8.2 below). Let $\tau^A = \{\tau^A_\alpha\}_{\alpha \in |\int A|}$ denote the colimit cocone, consisting of a family of morphisms $\tau^A_\alpha : \Delta_A(\alpha) \to \mathrm{colim}\Delta_A$, being automatically computations in $\mathrm{colim}\Delta_A$. Let δ_A denote here the projection function, taking a pair (P, p) to the computation p. An object A together with all its computations form the canonical cocone over Δ_A, which induces the unique morphism $!_A : \mathrm{colim}\Delta_A \to A$ making $\tau^A_\alpha ; !_A = \delta_A(\alpha)$, for all objects α in $\int A$. Consider the quotient $\overline{A}$ of $\mathrm{colim}\Delta_A$ by $\ker(|!_A|)$, with the quotient projection

$$\pi_A = \pi_{\mathrm{colim}\Delta_A/\ker(|!_A|)} : \mathrm{colim}\Delta_A \to \overline{A}.$$

Object $\overline{A}$ is necessarily a subobject of A with the injection $\iota_{\overline{A}\hookrightarrow A} = \overline{!_A}$; the function $|\iota_{\overline{A}\hookrightarrow A}|$ is necessarily injective, since the following diagram commutes:

$$\begin{tikzcd} & \overline{A} \arrow[d, hook, "\iota_{\overline{A}\hookrightarrow A}"] \\ \mathrm{colim}\Delta_A \arrow[ur, "\pi_A"] \arrow[r, "!_A"'] & A \end{tikzcd} \tag{19}$$

As $\mathbb{M}$ admits renaming of elements, we can assume without loss of generality that $|\overline{A}| \subseteq |A|$ and that $|\iota_{\overline{A}\hookrightarrow A}|$ is an inclusion.

Example 8.2 *The construction of the canonical colimit arises naturally in categories considered in previous sections.*

In the category of transition systems, operation $\mathrm{colim}\Delta\_$ *amounts to unfolding of a (reachable part of) transition system to a synchronization tree. For event structures, we have an even more pleasing situation – each event structure E is (up to isomorphism) equal to the canonical colimit* $\mathrm{colim}\Delta_E$*. Even more, Pom_L is a dense subcategory of event structures.*

In all the examples above, the inclusion $\iota_{\overline{A}\hookrightarrow A}$ is open – in fact this is always the case, and follows directly from the fact that categories $\int A$ and $\int \overline{A}$ are isomorphic (see the proof of Lemma 8.4 below). On the other hand, two other morphisms in the triangle (19) need not be open in general; instead, we formulate the following:

Observation 8.3 *Composition with $!_A$ transforms bijectively colimit inclusions τ^A_α to computations in A; moreover,*

$$\tau^A_\alpha = m;\tau^A_\beta \quad \textit{iff} \quad \tau^A_\alpha;!_A = m;\tau^A_\beta;!_A,$$

for any $\alpha, \beta \in |\int A|$ and any morphism $m : \Delta_A(\alpha) \to \Delta_A(\beta)$ in $\mathbb{P}$.

Now we are going to prove the fact analogous to the key Lemma 4.6, but without assumptions needed here (we only require $\mathbb{M}$ to be initialized) – besides this, the only difference is that subobjects $\widetilde{A}$ are replaced by $\overline{A}$.

Lemma 8.4 *For any objects A, B in $\mathbb{M}$ and any $\mathbb{P}$-open morphism $h : \overline{A} \to \overline{B}$, $\overline{B}$ is a quotient of $\overline{A}$, with the quotient projection h.*

Before embarking on the proof we need some more notation. Assumption 8.1 gives raise to an endofunctor $\mathcal{K} : \mathbb{M} \to \mathbb{M}$, taking an object A to $\mathrm{colim}\Delta_A$ and a morphisms $f : A \to B$ to $!_f : \mathrm{colim}\Delta_A \to \mathrm{colim}\Delta_B$ defined as follows. A morphism f induces a cocone ρ over Δ_A consisting of morphisms $\tau^A_\alpha;!_A;f$, for all α in $|\int A|$. But from this cocone we can construct another one, over the same diagram, consisting of precisely those morphisms β, having the codomain $\mathrm{colim}\Delta_B$, for which $\beta;!_B$ is in ρ. This cocone is unique by Observation 8.3 and gives a unique universal morphism $!_f : \text{colim}\Delta_A \to \text{colim}\Delta_B$. Moreover, by uniqueness of the universal morphism the following diagram commutes:

$$\begin{tikzcd}
\text{colim}\Delta_A \arrow[r, "!_A"] \arrow[d, "!_f"'] & A \arrow[d, "f"] \\
\text{colim}\Delta_B \arrow[r, "!_B"] & B
\end{tikzcd} \tag{20}$$

for all $f : A \to B$ in $\mathbb{M}$, that is to say the family $\{!_A\}_{A\in|\mathbb{M}|}$ is a natural transformation.

Remark Functor $\mathcal{K}$ defined above is actually the left Kan extension of the inclusion $\mathbb{P} \hookrightarrow \mathbb{M}$ along itself and is determined up to a natural isomorphism, as all colimits $\text{colim}\Delta_A$ are so.

Proof of Lemma 8.4: For a moment, let C be an arbitrary object in $\mathbb{M}$. By the construction of $\overline{C}$ it follows that $\iota_{\overline{C}\hookrightarrow C}$ is mono, which together with commutativity of (19) and Observation 8.3 implies that categories $\int C$ and $\int \overline{C}$ are isomorphic – an isomorphism is given by the composition with $\iota_{\overline{C}\hookrightarrow C}$. Since the composition with $\iota_{\overline{C}\hookrightarrow C}$ preserves a domain of a computation, we conclude that diagrams induced by Δ_C and $\Delta_{\overline{C}}$ are identical, hence $\text{colim}\Delta_C \cong \text{colim}\Delta_{\overline{C}}$. Moreover, a canonical isomorphism $i_C : \text{colim}\Delta_{\overline{C}} \to \text{colim}\Delta_C$ between both colimit objects, induced by the injection $\iota_{\overline{C}\hookrightarrow C}$ (i.e., $i_C = !_{\iota_{\overline{C}\hookrightarrow C}}$ according to the notation from (20) above), factors $!_{\overline{C}}$ through π_C. Summing up, both triangles commute in the following diagram:

$$\begin{tikzcd}
\text{colim}\Delta_{\overline{C}} \arrow[r, "!_{\overline{C}}"] \arrow[d, "i_C"'] & \overline{C} \arrow[d, hook, "\iota_{\overline{C}\hookrightarrow C}"] \\
\text{colim}\Delta_C \arrow[r, "!_C"'] \arrow[ur, "\pi_C"] & C
\end{tikzcd} \tag{21}$$

Let $h : \overline{A} \to \overline{B}$ be an open morphism in $\mathbb{M}$. By combining two copies of diagram (21), for A and B, respectively, with an instantiation of diagram (20) for h, we obtain the following:

$$\begin{tikzcd}
\text{colim}\Delta_{\overline{A}} \arrow[r, "!_{\overline{A}}"] \arrow[d, "i_A"'] \arrow[drr, "!_h"] & \overline{A} \arrow[d, hook, dotted, "\iota_{\overline{A}\hookrightarrow A}"] \arrow[drr, "h"] & & \\
\text{colim}\Delta_A \arrow[r, dotted, "!_A"] \arrow[ur, dotted, "\pi_A"] \arrow[drr, "{i_A}^{-1};!_h;i_B"'] & A & \text{colim}\Delta_{\overline{B}} \arrow[r, "!_{\overline{B}}"] \arrow[d, "i_B"'] & \overline{B} \arrow[d, hook, "\iota_{\overline{B}\hookrightarrow B}"] \\
& & \text{colim}\Delta_B \arrow[r, "!_B"'] \arrow[ur, "\pi_B"] & B
\end{tikzcd} \tag{22}$$

from which we choose the square determined by two diagonals π_A and π_B:

$$\begin{tikzcd}
\text{colim}\Delta_A \arrow[r, "\pi_A"] \arrow[d, "k:={i_A}^{-1};!_h;i_B"'] & \overline{A} \arrow[d, "h"] \\
\text{colim}\Delta_B \arrow[r, "\pi_B"] & \overline{B}
\end{tikzcd} \tag{23}$$

It necessarily commutes, since all squares commute in (22). We begin by showing that

1. k is epi, and
2. k is a quotient projection.

(1) As h is open and $\mathbb{M}$ is initialized, for every computation $(p; !_{\overline{B}})$ in $\overline{B}$ (with p a cocone injection) there exists another computation $(p'; !_{\overline{A}})$ in $\overline{A}$ (with p' a cocone injection, again) such that $p'; !_{\overline{A}}; h = p; !_{\overline{B}}$:

$$\begin{tikzcd} & \mathrm{colim}\Delta_{\overline{A}} \arrow[r, "!_{\overline{A}}"] & \overline{A} \arrow[d, "h"] \\ {} \arrow[ur, "p'"] \arrow[urr, dotted] \arrow[r, "p"] & \mathrm{colim}\Delta_{B} \arrow[r, "!_{\overline{B}}"] & \overline{B} \end{tikzcd}$$

(we implicitly used here Observation 8.3). The upper side of diagram (22) commutes, hence $p'; !_h; !_{\overline{B}} = p; !_{\overline{B}}$. Moreover, since the composition with $!_{\overline{B}}$ cannot identify different cocone computations in $\mathrm{colim}\Delta_B$ by Observation 8.3, we conclude that for every cocone computation p in $\mathrm{colim}\Delta_{\overline{B}}$ there exists another cocone computation p' in $\mathrm{colim}\Delta_{\overline{A}}$ meeting $p'; !_h = p$. This statement can be "translated" through isomorphisms i_A and i_B as follows: for every cocone computation p in $\mathrm{colim}\Delta_B$ there exists another cocone computation p' in $\mathrm{colim}\Delta_A$ meeting $p'; k = p$.

Now assume $k; f = k; g$, from which it follows $p'; (k; f) = p'; (k; g)$, for all computations p' in $\mathrm{colim}\Delta_A$; this implies $p; f = p; g$ for all computations p in $\mathrm{colim}\Delta_B$, hence $f = g$.

(2) Consider any morphism $f : \mathrm{colim}\Delta_A \to D$ such that $\ker(|k|) \subseteq \ker(|f|)$. It is easy to observe that D together with morphisms $(p'; f)$, for all cocone computations p' in $\mathrm{colim}\Delta_A$, form a cocone over Δ_B. It is so, since for any pair of cocone computations p_1, p_2 in $\mathrm{colim}\Delta_B$ satisfying $m; p_1 = p_2$ (where m is a morphism in $\mathbb{P}$), we have two cocone computations p'_1, p'_2 in $\mathrm{colim}\Delta_A$ satisfying $m; p'_1; k = p'_2; k$, from which we conclude $m; (p'_1; f) = (p'_2; f)$. The described cocone induces the unique morphism $g : \mathrm{colim}\Delta_B \to D$ satisfying $p; g = p'; f$, for all cocone computations p, where p' is a corresponding cocone computation satisfying $p'; k = p$. Hence we obtain $p'; (k; g) = p'; f$, for all cocone computations p' in $\mathrm{colim}\Delta_A$, from which we conclude $k; g = f$ (and such g is unique, since k is epi), which completes the proof of (2).

Now we should show that h is the quotient projection. By (1) h is epi, as diagram (23) commutes and $(\pi_A; h) = (k; \pi_B)$ is epi, which follows from the fact that k is epi and epimorphisms compose. By (2) we conclude that $(\pi_A; h) = (k; \pi_B)$ is a quotient projection, since surjective quotient projections compose (this is routine to verify). Now we are able to prove that h is a quotient projection. Let $f : \overline{A} \to C$ be a morphism with $\ker(|h|) \subseteq \ker(|g|)$. As $|\pi_A|$ is surjective, we have $\ker(|\pi_A; h|) \subseteq \ker(|\pi_A; f|)$, hence there exists a (unique) morphism $\bar{f} : \overline{B} \to C$ making $(\pi_A; h); \bar{f} = (\pi_A; f)$. Now it easily follows from π_A being epi that $h; \bar{f} = f$, which completes the proof. $\square$

Remark In the proof of (1) we actually argued that k, being the image of $\mathcal{K}$ on an open morphism h (up to isomorphisms i_A, i_B), is open itself. This is related to the result of Cattani and Winskel [CW96], stating that all left Kan extensions along Yoneda embeddings between presheaf categories preserve openness. Notice that we showed more than this – namely that k is a surjective quotient projection.

8.1 Relating domain subobjects

The technical problem concerned with constructing domain subobjects from colimits of computations is that $\overline{A}$ need not to be a full subobject of A, for example for empty subcategory $\mathbb{P}$. Nevertheless, some links between $\overline{A}$ and $\widetilde{A}$ can be established. We start by relating the carrier sets:

Lemma 8.5 $\mathbb{COMP}_A \subseteq |\overline{A}| \subseteq |\widetilde{A}|$, *for each object* A *in* $\mathbb{M}$.

Proof: In the proof of Lemma 8.4 we showed that categories $\int A$ and $\int \overline{A}$ are isomorphic; this together with injectivity of the inclusion $\iota_{\overline{A}\hookrightarrow A}$ implies the first inclusion $\mathbb{COMP}_A \subseteq |\overline{A}|$. To prove the second one, notice that $\widetilde{A}$ together with all computations in $\widetilde{A}$ forms a cocone over Δ_A, which follows from $\widetilde{A}$ being the full subobject. Since $\mathbb{M}$ is initialized, by openness and monotonicity of the subobject inclusion $\iota_{\widetilde{A}\hookrightarrow A}$, categories of computations $\int A$ and $\int \widetilde{A}$ are necessarily isomorphic. Thus the induced universal morphism $g : \mathrm{colim}\Delta_A \to \widetilde{A}$ satisfies $\tau^A_\alpha;(g;\iota_{\widetilde{A}\hookrightarrow A}) = \delta_A(\alpha)$, for all α in $|\int A|$. From this we conclude that $g;\iota_{\widetilde{A}\hookrightarrow A} = !_A$, i.e., $g;\iota_{\widetilde{A}\hookrightarrow A} = \pi_A;\iota_{\overline{A}\hookrightarrow A}$:

$$\begin{tikzcd}
 & \overline{A} \arrow[d, hook, "\iota_{\overline{A}\hookrightarrow A}"] \\
\mathrm{colim}\Delta_A \arrow[ur, "\pi_A"] \arrow[r, "!_A"'] \arrow[d, "g"'] & A \\
\widetilde{A} \arrow[ur, hook, "\iota_{\widetilde{A}\hookrightarrow A}"'] &
\end{tikzcd} \tag{24}$$

As both $|\iota_{\widetilde{A}\hookrightarrow A}|$ and $|\iota_{\overline{A}\hookrightarrow A}|$ are injective and $|\pi_A|$ is surjective, we obtain $|\overline{A}| \subseteq |\widetilde{A}|$. □

From this lemma it follows that whenever $\overline{A}$ is full, it necessarily coincides with $\widetilde{A}$. Moreover, both subobjects coincide, when the assumptions from Theorem 4.4 are fulfilled:

Proposition 8.6 *When* $|\_|$ *lifts isomorphisms of domain subobjects and computations create subobjects in* $\mathbb{M}$, *then* $\overline{A} \cong \widetilde{A}$.

Proof: By the previous lemma, there exists an injective morphism $\xi : \overline{A} \to \widetilde{A}$, given by $\xi := \widehat{\iota_{\overline{A}\hookrightarrow A}}$ in diagram (24).

$$\begin{tikzcd}
 & \overline{A} \arrow[dl, "\xi"'] \arrow[d, hook, "\iota_{\overline{A}\hookrightarrow A}"] \\
\widetilde{A} \arrow[r, hook, "\iota_{\widetilde{A}\hookrightarrow A}"'] & A
\end{tikzcd}$$

Now, knowing that $\mathbb{COMP}_A = |\widetilde{A}|$, we conclude that ξ is also surjective. Moreover, ξ is open by Proposition 4 since $(\xi;\iota_{\widetilde{A}\hookrightarrow A}) = \iota_{\overline{A}\hookrightarrow A}$ is open and $\iota_{\widetilde{A}\hookrightarrow A}$ is mono. And finally, by the assumption that $|\_|$ lifts isomorphisms of domain subobjects we conclude that ξ is iso; since $\mathbb{M}$ admits renaming of elements, this completes the proof. □

As a conclusion from the above considerations, Lemma 8.4 can be used in Theorem 4.4 only when all the assumptions needed there are satisfied. Hence, the colimit construction does not enable, in general, to drop any of assumptions from Theorem 4.4.

9 Final remarks

In the paper we analyzed conditions sufficient for the factorization of bisimilarity by partial open congruences. This general result gives an interesting conclusion that a wide class of bisimilarities (including those for transition systems, event structures and presheaves) can be equivalently defined by co-spans of open quotient projections, similarly as algebraic behavioural equivalences. We also found the necessary and sufficient condition for the factorization: quotients by all maximal open congruences are isomorphic.

The natural continuation of this work is to check whether the results can be applied to other equivalences of processes, like testing equivalence or weak bisimilarity (captured in [CN95] in the setting of open maps).

Our approach offers still some potentials for future investigations, for instance Lemmas 2.1 and 7.4 suggest some general line of establishing coincidence between bisimulations and open congruences, giving automatically the greatest open congruences. Another challenge is to generalize the proof of Theorem 6.7, specific for event structures, possibly under an assumption that open quotients are split.

The crucial role is played in our development by the subobjects $\widetilde{A}$ generated by images of computations. In the last section we presented another possibility to define a domain subobject of A, as a canonical colimit of all computations in A. This construction, being more natural in category theory, ensures without any special assumptions that open maps between domain subobjects are quotient projections (the key fact needed in the proof of Theorem 4.4). Unfortunately, the price to pay is that these subobjects are not necessarily full. They do in all examples considered, since under assumptions from Theorem 4.4 they coincide with subobjects generated by images of computations.

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References

[AO93] S. Abramsky, C.-H. Luke Ong, Full Abstraction in the Lazy Lambda Calculus. Information and Computation 105(2) (1993) 159-267.

[Bed91] M. Bednarczyk, Hereditary history preserving bisimulations or what is the power of future perfect in programs logics. Technical report, Polish Academy of Sciences, Gdańsk (1991).

[BB88] D.B. Benson, O. Ben-Schachar, Bisimulation of automata. *Information and Computation*, 79 (1988) 60-83.

[BHW95] M. Bidoit, R. Hennicker, M. Wirsing, Behavioural and abstractor specifications. Science of Computer Programming 25(2-3) (1995) 149-186.

[BT95] Bidoit, M., Tarlecki, A. Regular algebras: a framework for observational specifications with recursive definitions. Report LIENS-95-12, Ecole Normale Superieure (1995).

[BT96] Bidoit, M., Tarlecki A. Behavioural satisfaction and equivalence in concrete model categories. Proc. 20th Coll. on Trees in Algebra and Computing CAAP'96, Linköping, LNCS 1059 (Springer-Verlag, 1996) 241-256.

[C85] Castellani, I. Bisimulation and abstraction homomorphisms. Proc. CAAP'85, LNCS 186 (Springer-Verlag, 1985) 223-238.

[CW96] G.L. Cattani, G. Winskel, Presheaf models for concurrency. Proc. Computer Science Logic CSL'96, LNCS 1258 (Springer-Verlag, 1997) 58-75. Preliminary version appeared as BRICS Report RS-96-35.

[CN95] A. Cheng, M. Nielsen, Open maps (at) work. Research series RS-95-23, BRICS, Department of Comp. Sc., University of Aarhus (1995).

[DNM89] P. Degano, R. de Nicola, U. Montanari, Partial orderings descriptions and observations of nondeterministic concurrent processes, in: Linear Time, Branching Time and Partial Order in Logics and Models for Concurrency, LNCS 354 (Springer-Verlag, 1989), 438-466.

[GGM76] V. Giarratana, F. Gimona, U. Montanari, Observability concepts in abstract data type specification. Proc. 5th Intl. Symp. Mathematical Foundations of Computer Science, Gdańsk 1976, LNCS 45 (Springer-Verlag, 1976) 576-587.

[vGG89] R.J. Van Glabeek, U. Goltz, Equivalence notions for concurrent systems and refinement of actions. Proc. MFCS'89, LNCS 379 (Springer-Verlag, 1989) 237-248.

[Gor95] Gordon, A. A tutorial on co-induction and functional programming. Proc. the 1994 Glasgow Workshop on Functional Programming, Springer Workshops in Computing, 1995, 78-95.

[JM94] A. Joyal, I. Moerdijk, A completeness theorem for open maps. Annals of Pure and Applied Logic 70 (1994) 51-86.

[JNW93] A. Joyal, M. Nielsen, G. Winskel, Bisimulation and open maps. Proc. 8th Annual Symposium on Logic in Computer Science LICS'93 (1993) 418-427.

[Las97] S. Lasota, Open Maps as a Bridge between Algebraic Observational Equivalence and Bisimilarity. Proc. 12th Workshop on Algebraic Developement Techniques, Tarquinia, LNCS 1376 (Springer-Verlag, 1997) 285-299.

[Las98] S. Lasota, Partial-Congruence Factorization of Bisimilarity Induced by Open Maps. Proc. 25th ICALP, Aalborg, LNCS 1443 (Springer-Verlag, 1998) 91-102.

[Miln89] R. Milner, Communication and concurrency. Prentice-Hall International Series in Computer Science, C. A. R. Hoare series editor (1989).

[NW95] M. Nielsen, G. Winskel, Models for concurrency, in: eds. S. Abramsky, D. M. Gabbay and T.S. Gabbay, The Handbook of Logic in Computer Science, vol. 4 (Oxford University Press, 1995).

[Par81] D.M.R. Park, Concurrency and Automata on Infinite Sequences. Proc. 5th G.I. Conference, LNCS 104 (Springer-Verlag, 1981).

[RT88] A. Rabinovich, B. Trakhtenbrot, Behaviour structures and nets of processes. Fundamenta Informaticae, 11:357-404 (1988).

[Rei81] H. Reichel, Behavioural equivalence – a unifying concept for initial and final specification methods. Proc. 3rd Hungarian Computer Science Conference, Mathematical Models in Computer Systems, M. Arato, L. Varga, eds., Budapest (1981) 27-39.

[Win99] G. Winskel, Event Structures as Presheaves – Two Representation Theorems. BRICS Report RS-99-7 (1999).