Computing Measure of Regular Languages of Infinite Trees

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jointly with Damian Niwiński, Paweł Parys, and Marcin Przybyłko

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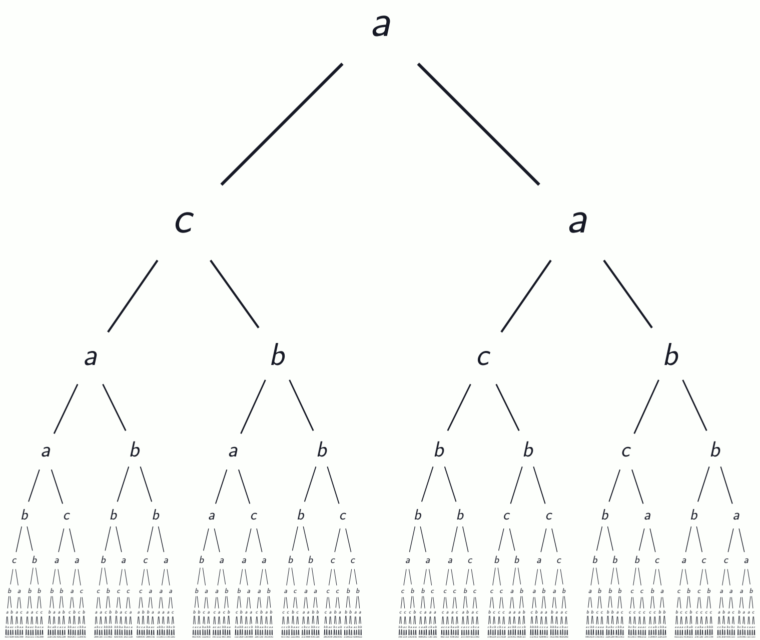

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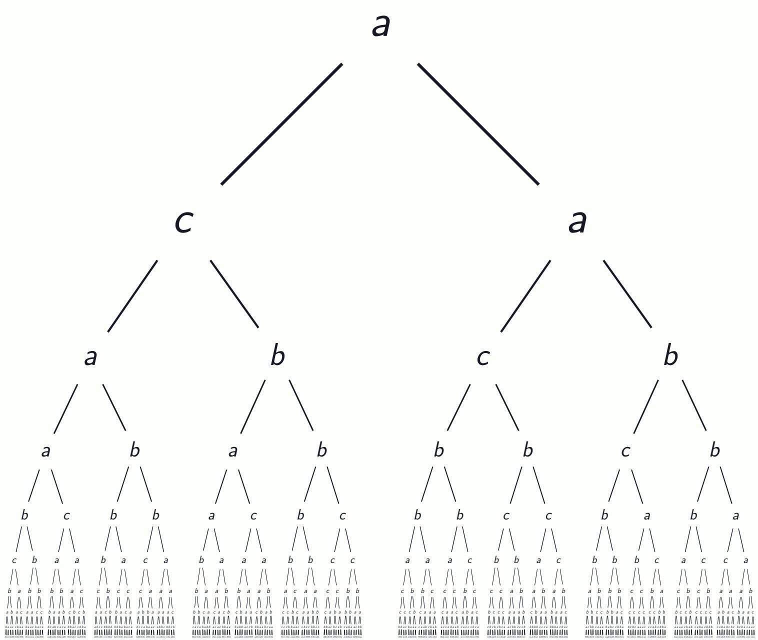

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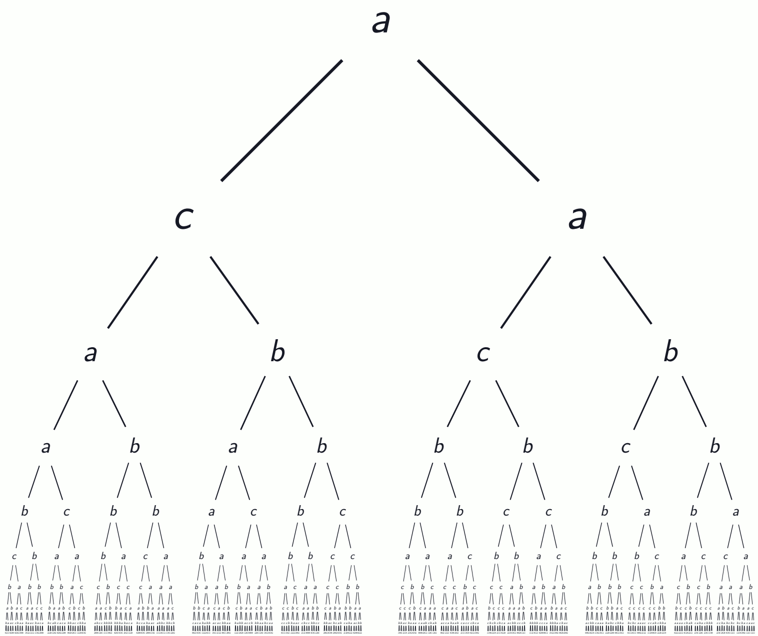

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Theorem (Gogacz, Michalewski, Mio, S. [2017])

Every regular language of infinite trees is **measurable**.

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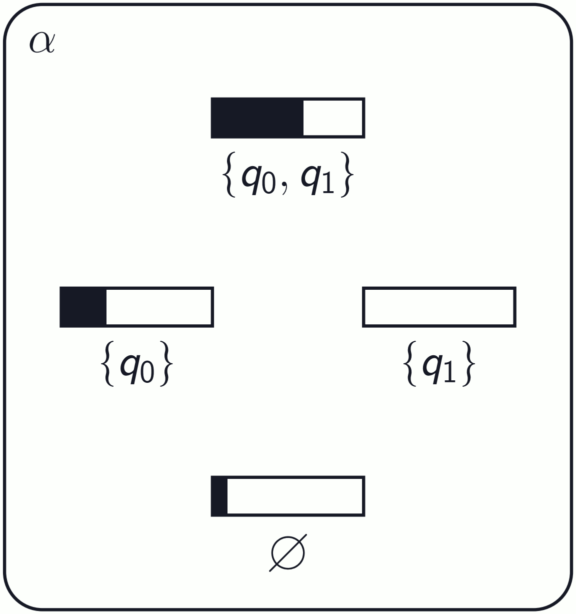

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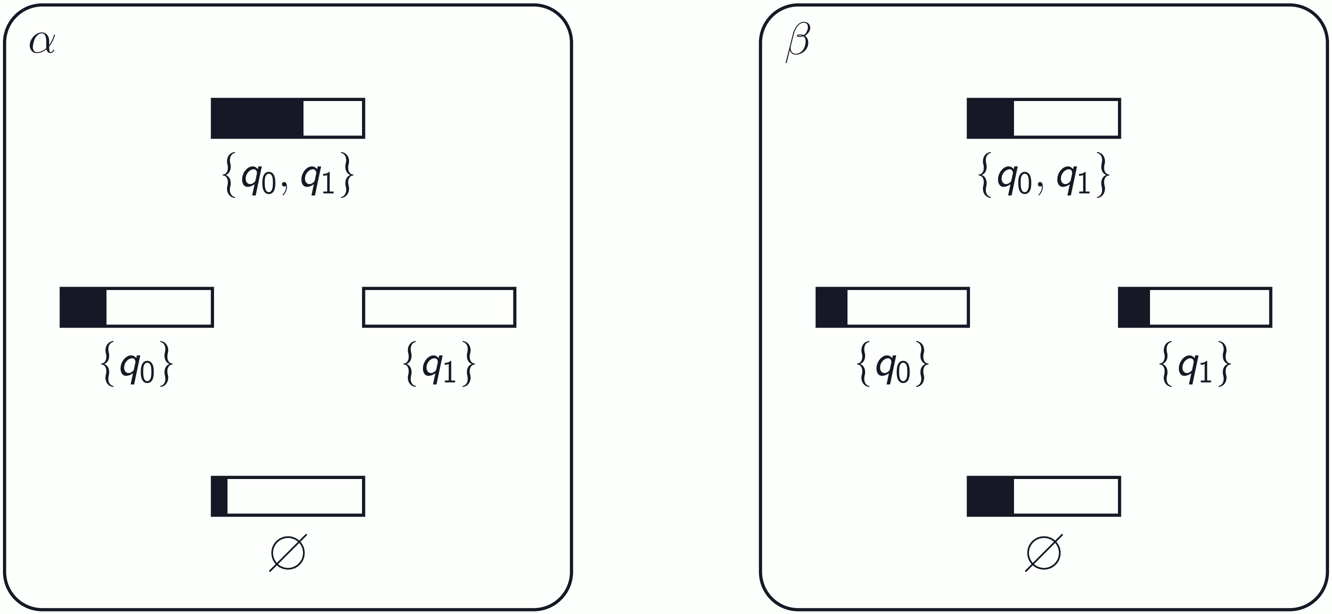

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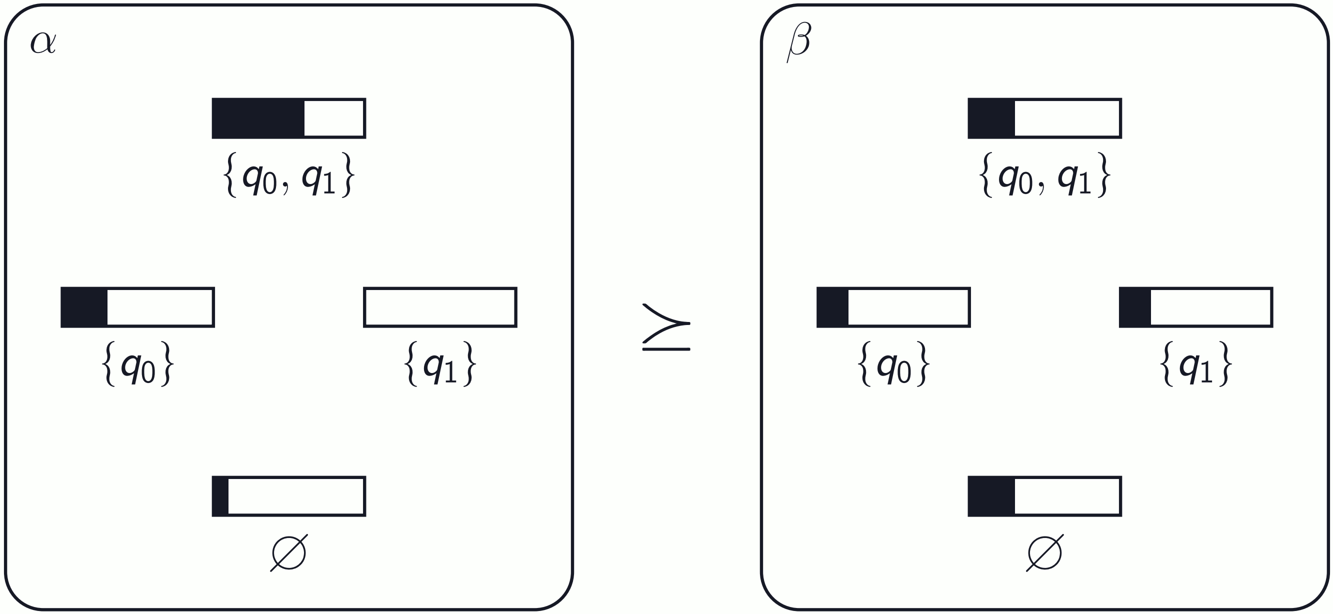

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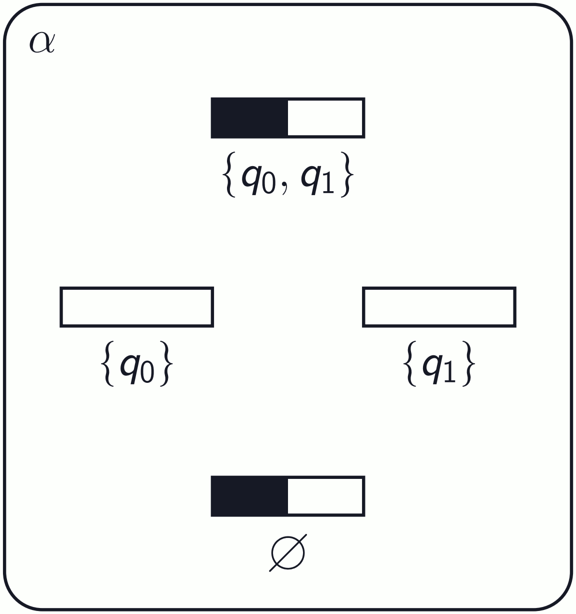

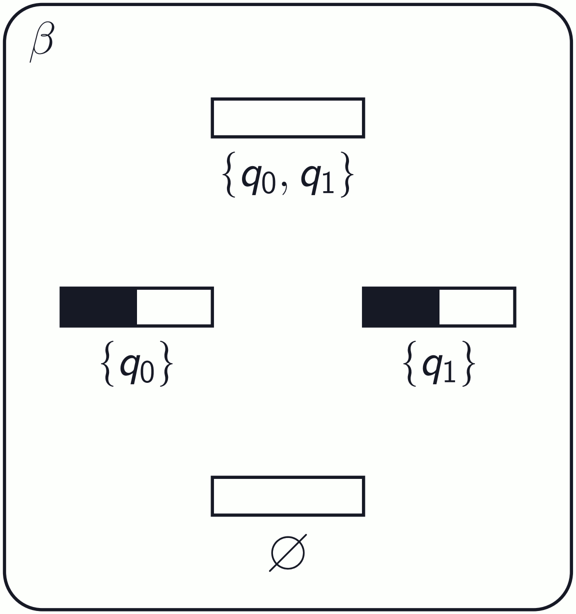

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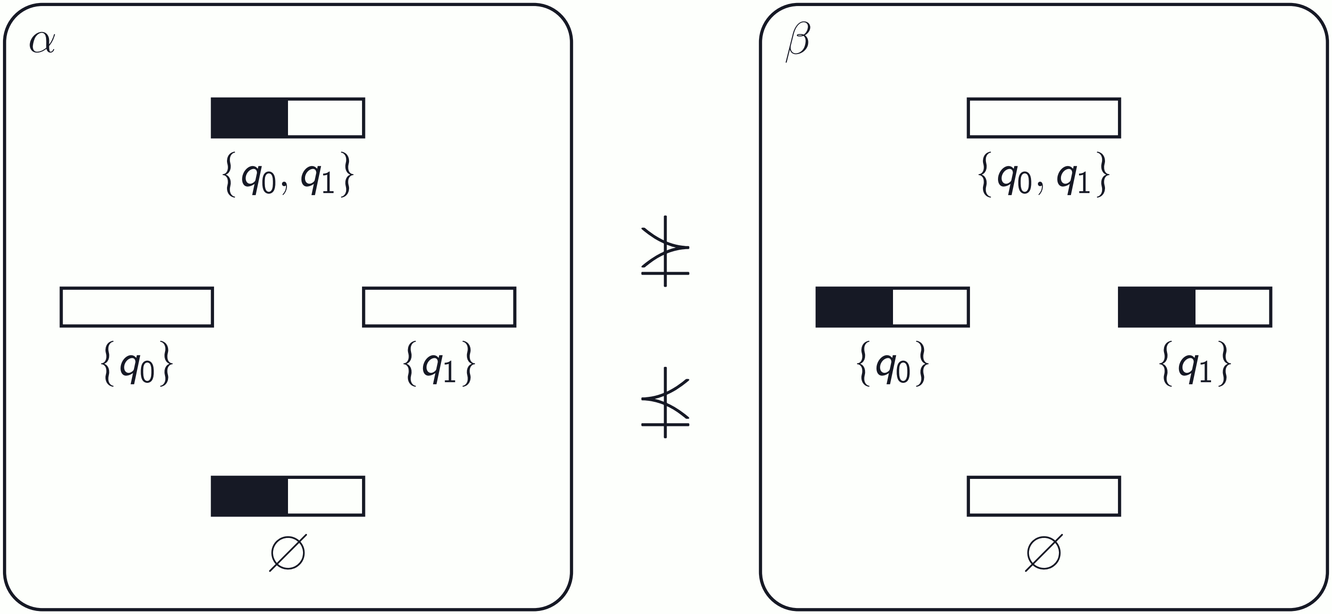

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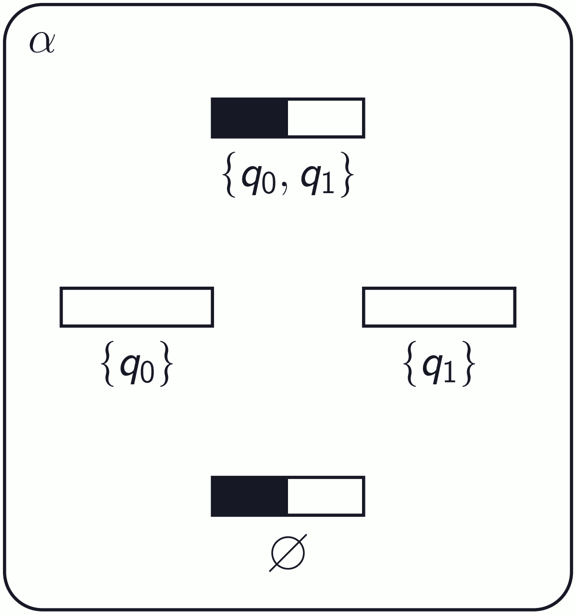

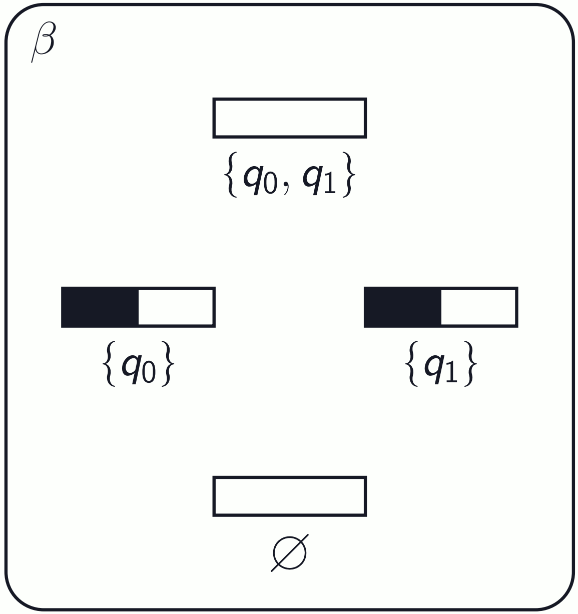

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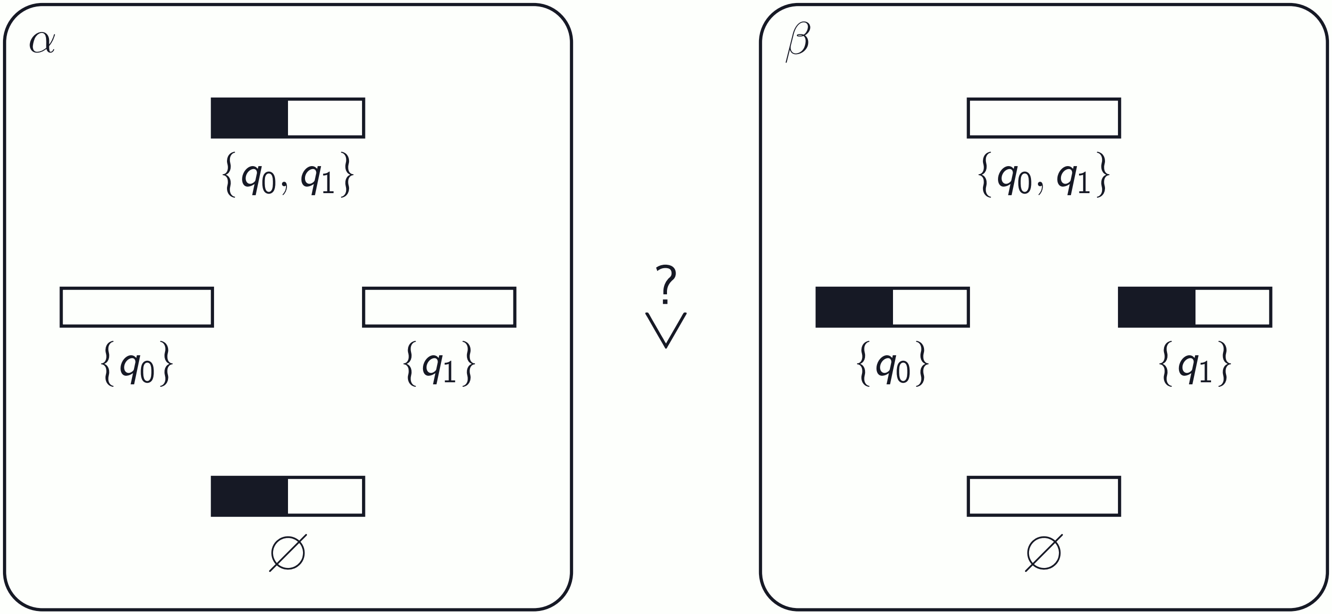

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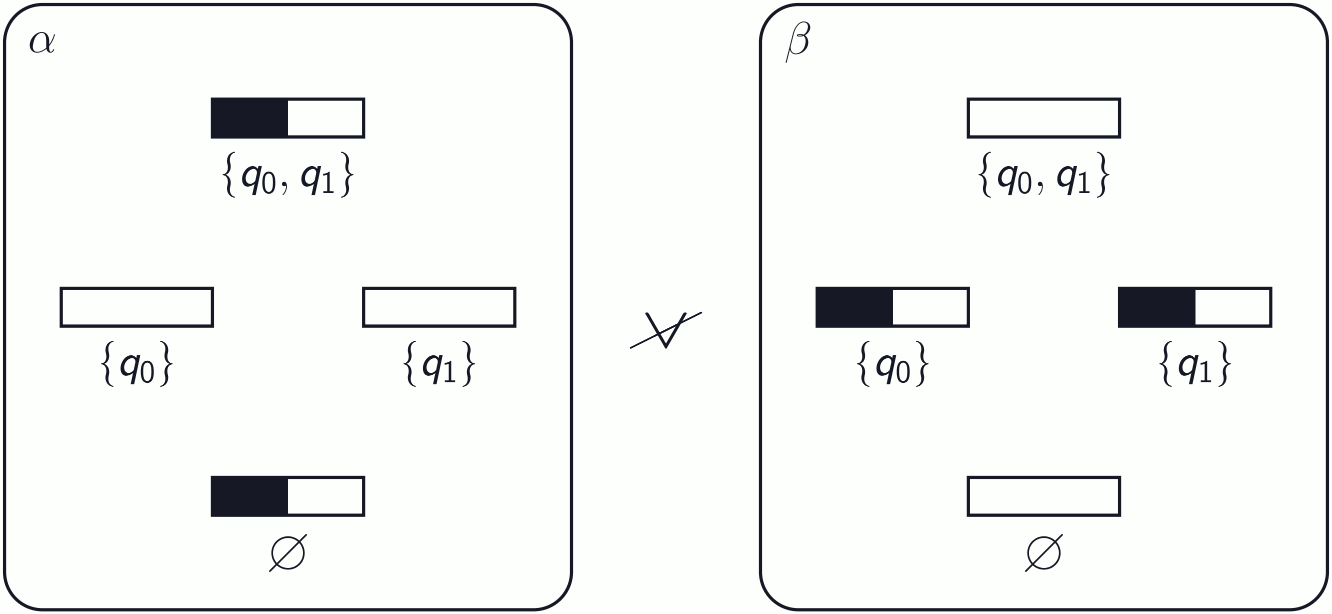

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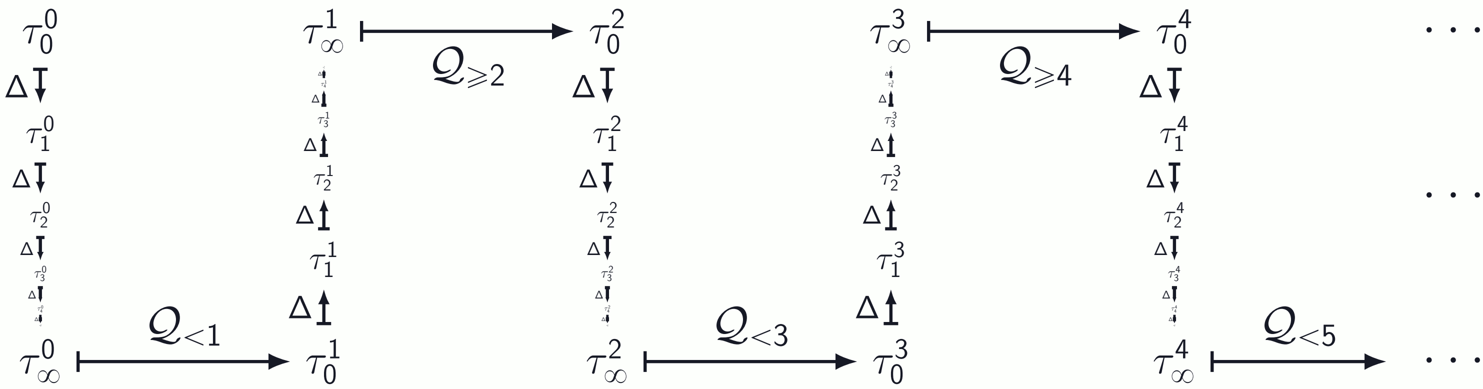

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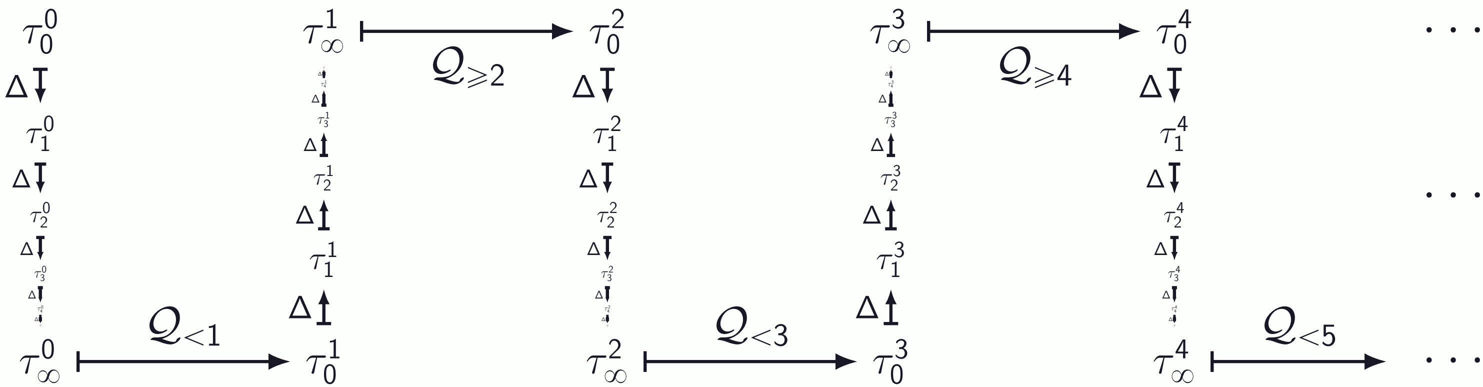

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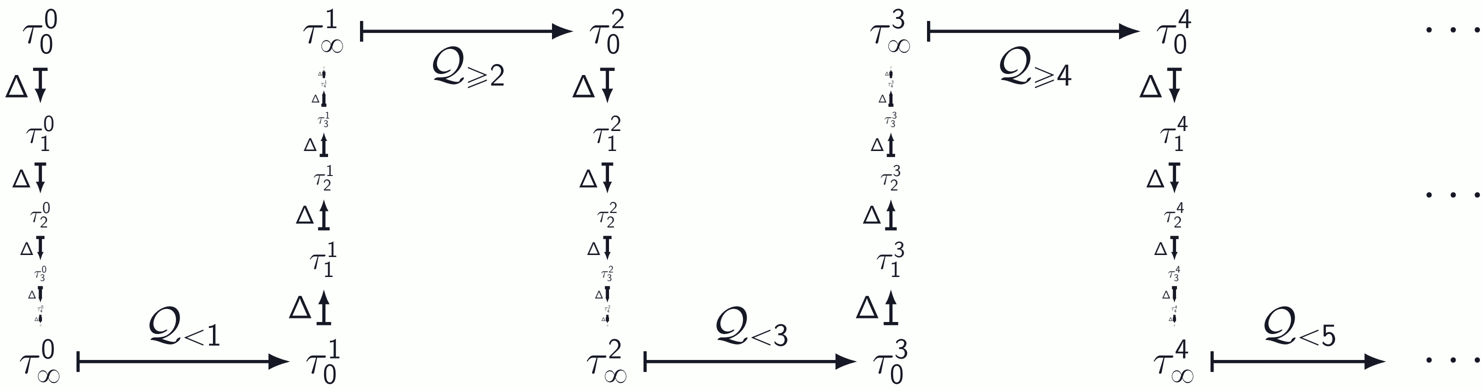

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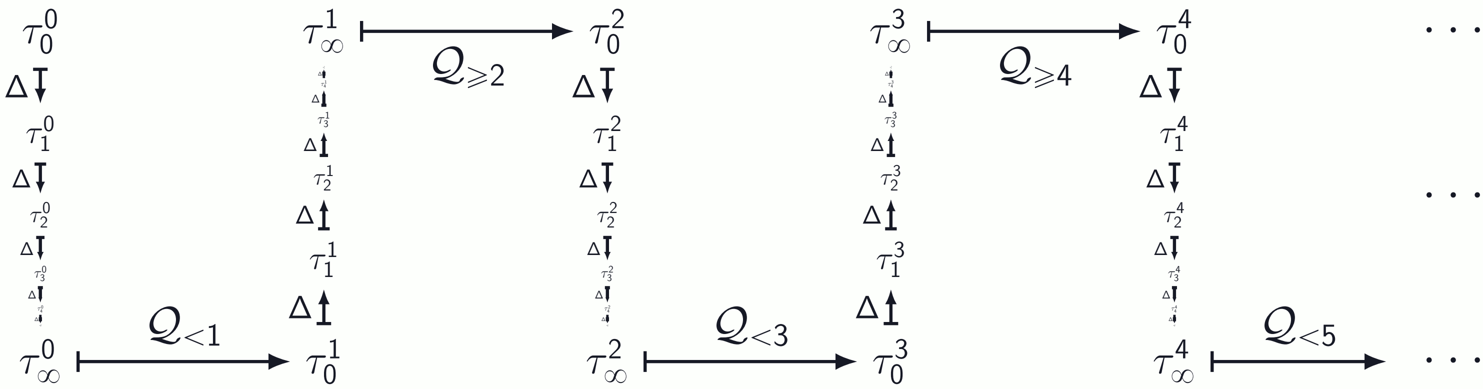

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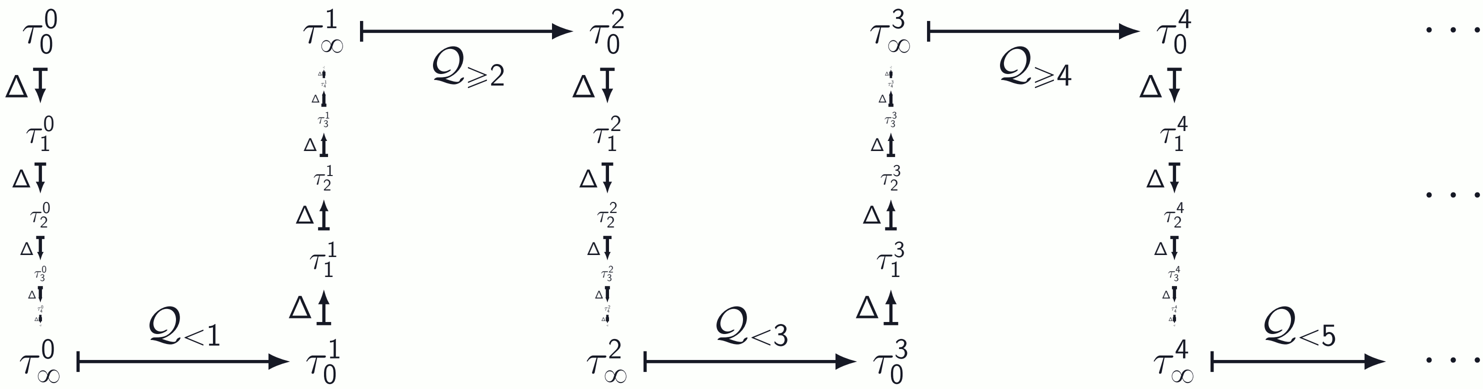

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$$\mathsf{P}\left(\mathrm{Tr}_A \times Q\right)$$

3. Prove that the result equals

$$\mathrm{tp}_{\mathcal{A}} \colon \mathrm{Tr}_A \to \mathsf{P}(Q)$$

4. Interpret it over distributions, *i.e.* in

$$\mathcal{D}\big(\mathsf{P}(Q)\big)$$

5. The result necessarily is

$$\overline{\mathrm{tp}_{\mathcal{A}}} \in \mathcal{D}(\mathsf{P}(Q))$$

6. Compute the result in Tarki's First-order theory of

$$\mathbb{R}^{\mathsf{P}(Q)}$$

⇝ Obtain $\mathbb{P}\left(\mathrm{L}(\mathcal{A})\right)$ as an algebraic number!