Measure theory and monadic second-order logic over infinite trees

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Infinite trees

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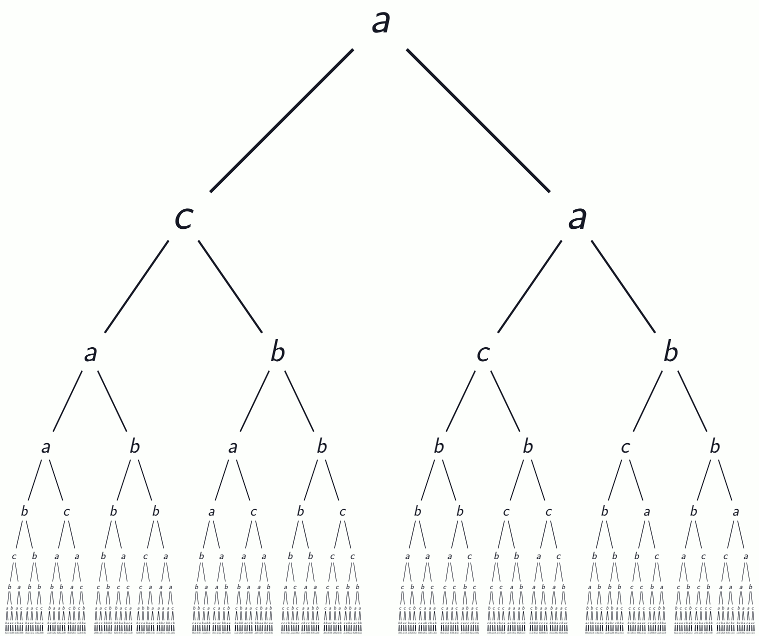

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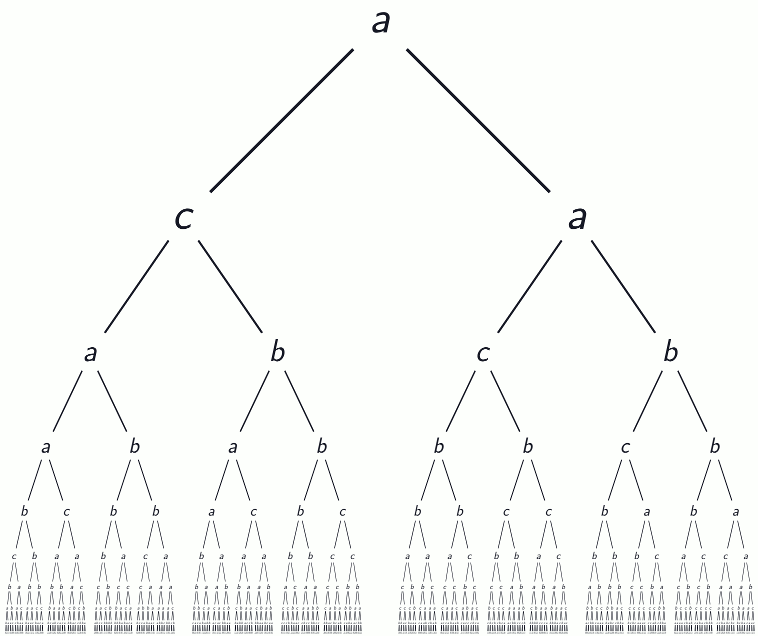

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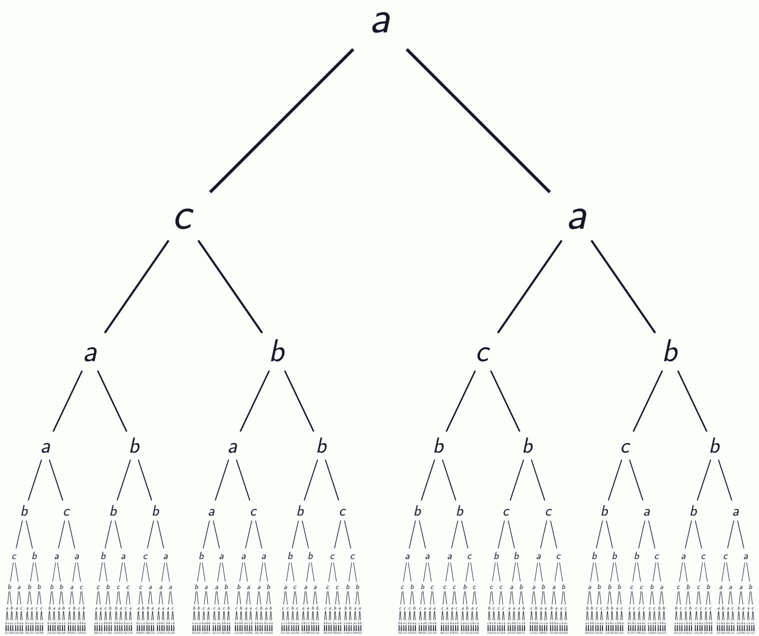

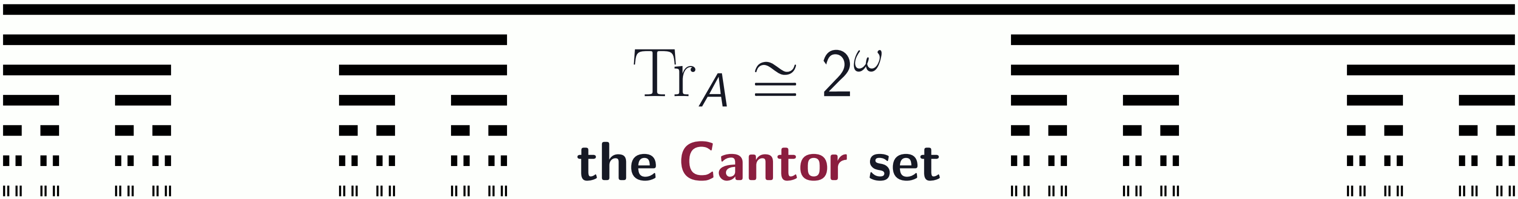

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the Cantor set

Monadic second-order logic (**mso**):

$$\varphi ::= \exists x.\ \varphi \mid \varphi \vee \varphi \mid \neg\varphi \mid x = y \mid \exists X.\ \varphi \mid x \in X \mid x \preceq y \mid x \leqslant_{\mathrm{lex}} y \mid a(x)$$

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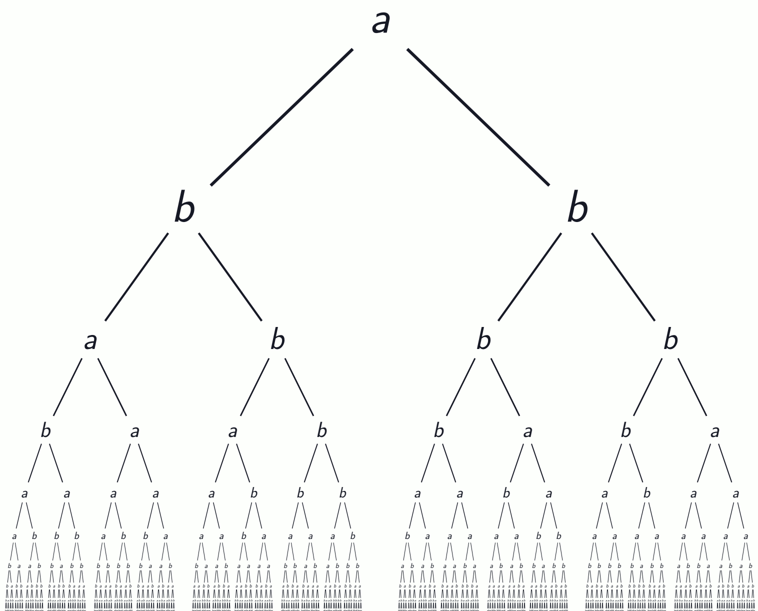

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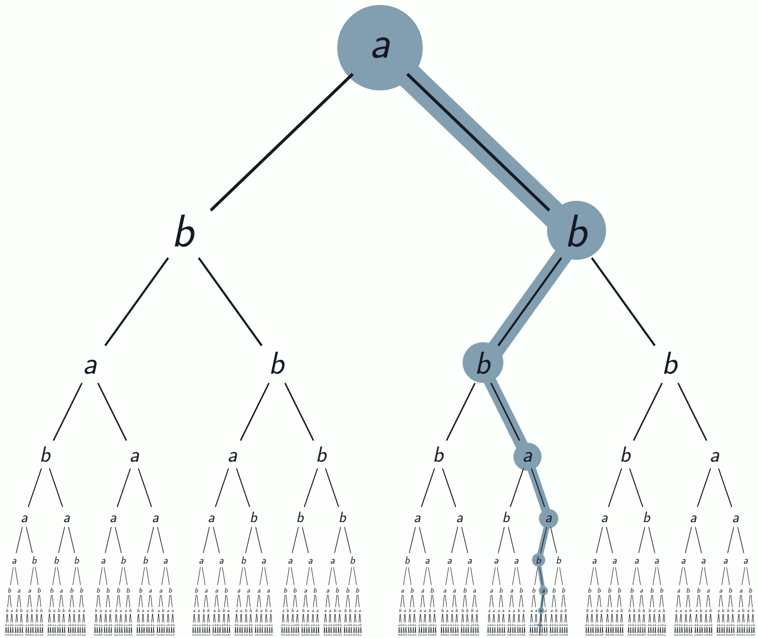

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Theorem (Rabin [1969])

The MSO theory of infinite trees is **decidable**:

INPUT: φ

OUTPUT: $\mathrm{L}(\varphi) \stackrel{?}{\neq} \emptyset$

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"coin-flipping measure" (for an A-sided coin)

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(Potthoff [1994])

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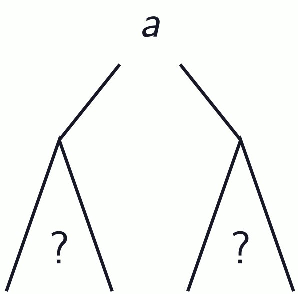

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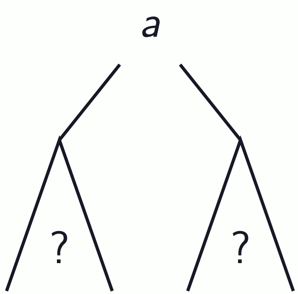

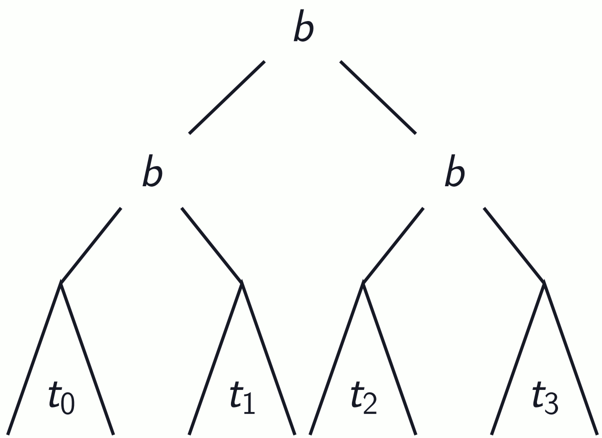

$t_0, t_1, t_2, t_3 \in \mathrm{L}(\varphi)$

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$$x_0 = 0.508347\ldots \qquad x_1 = 1.79358\ldots$$

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$$\varphi = \forall B.\ \mathrm{branch}(B) \Rightarrow \exists v \in B.\ a(v) \wedge \left(\forall u < v.\ b(u)\right) \wedge \mathrm{even}(v)$$

"the first a on every branch is at an even depth" $\qquad A = \{a, b\}$

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mso ≡ Alternating parity tree automata

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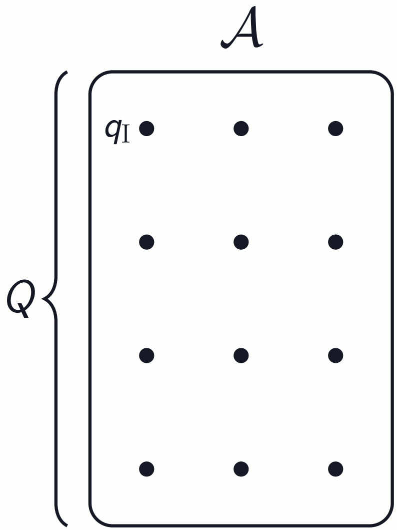

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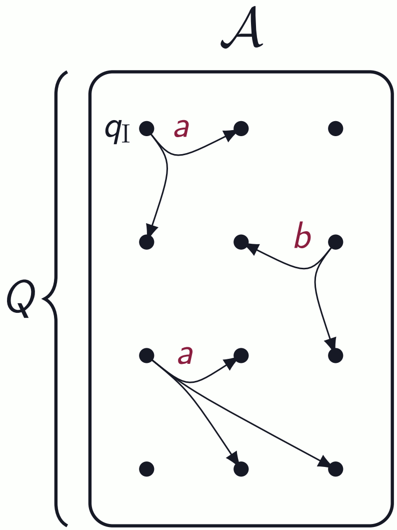

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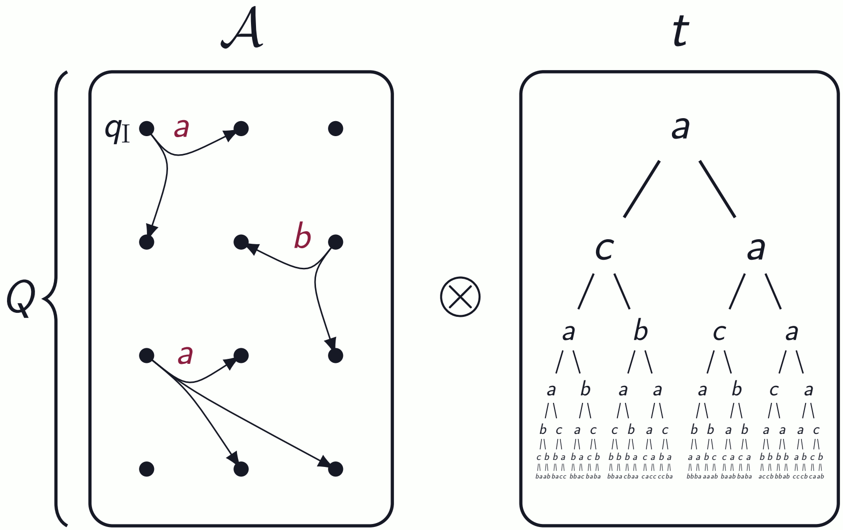

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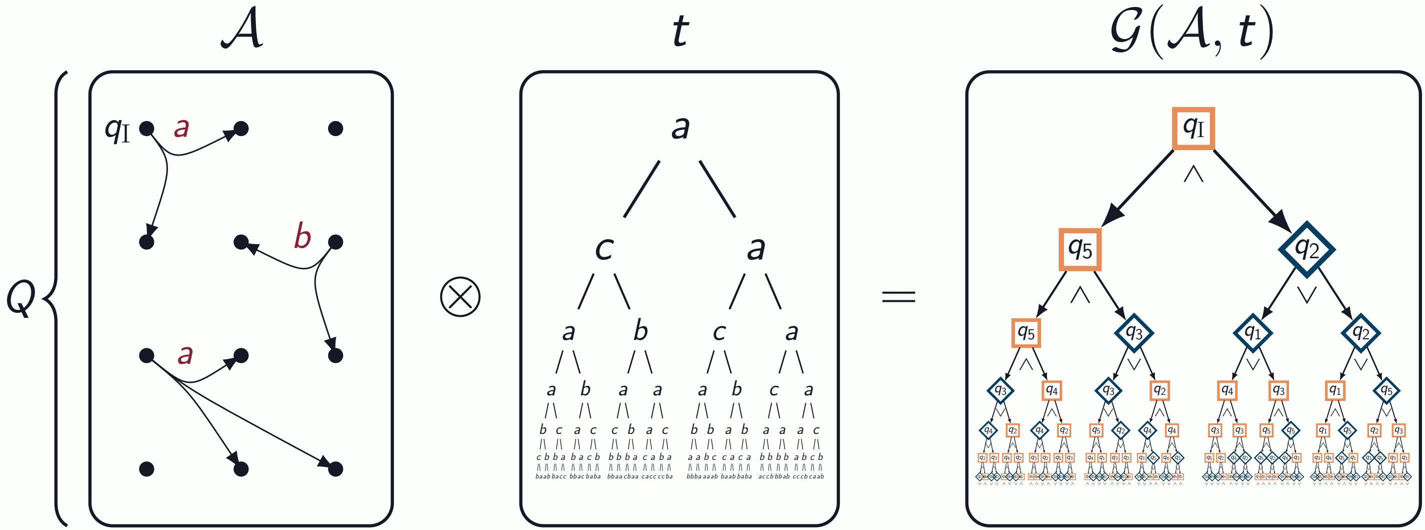

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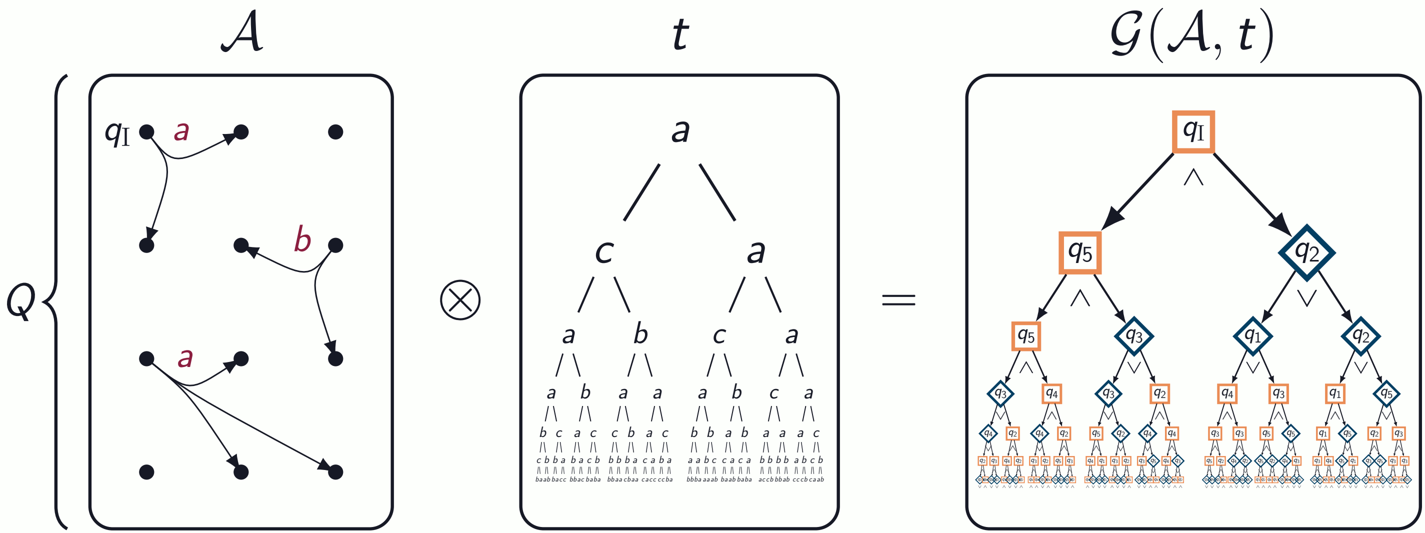

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Game played between ◇ and □

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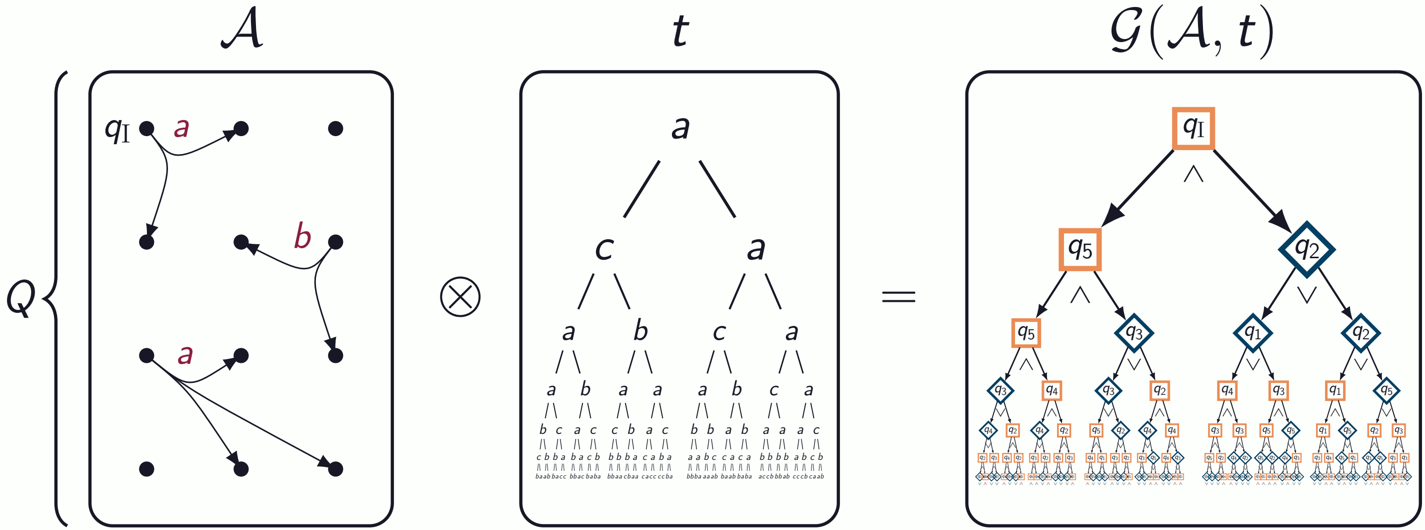

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Parity condition: $\Omega\colon Q \to \mathbb{N}$

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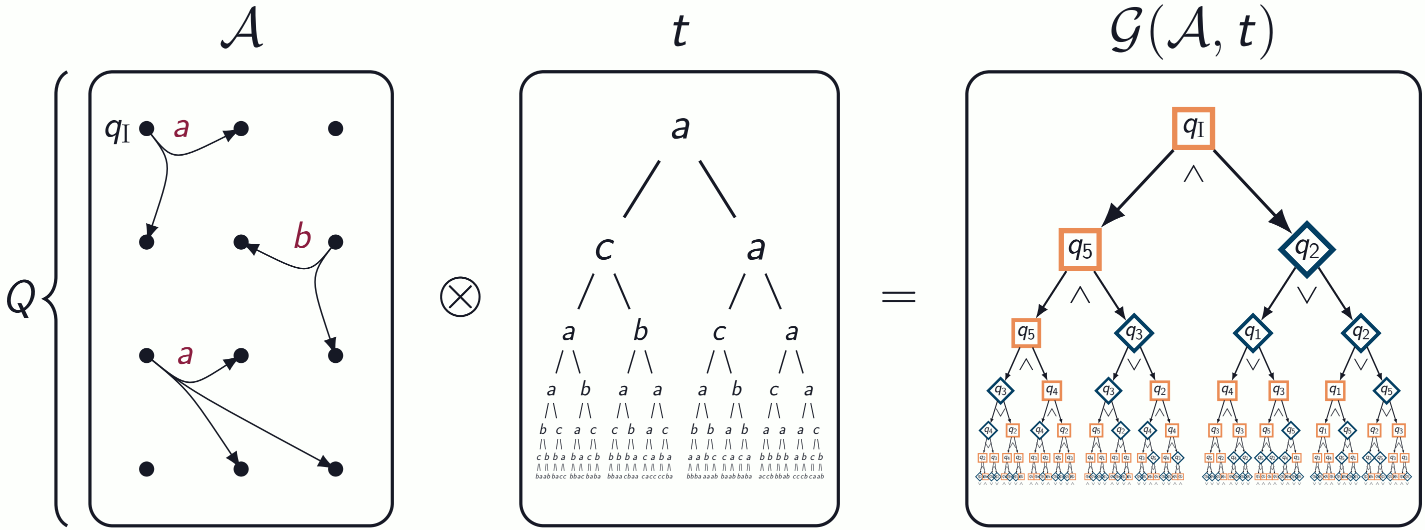

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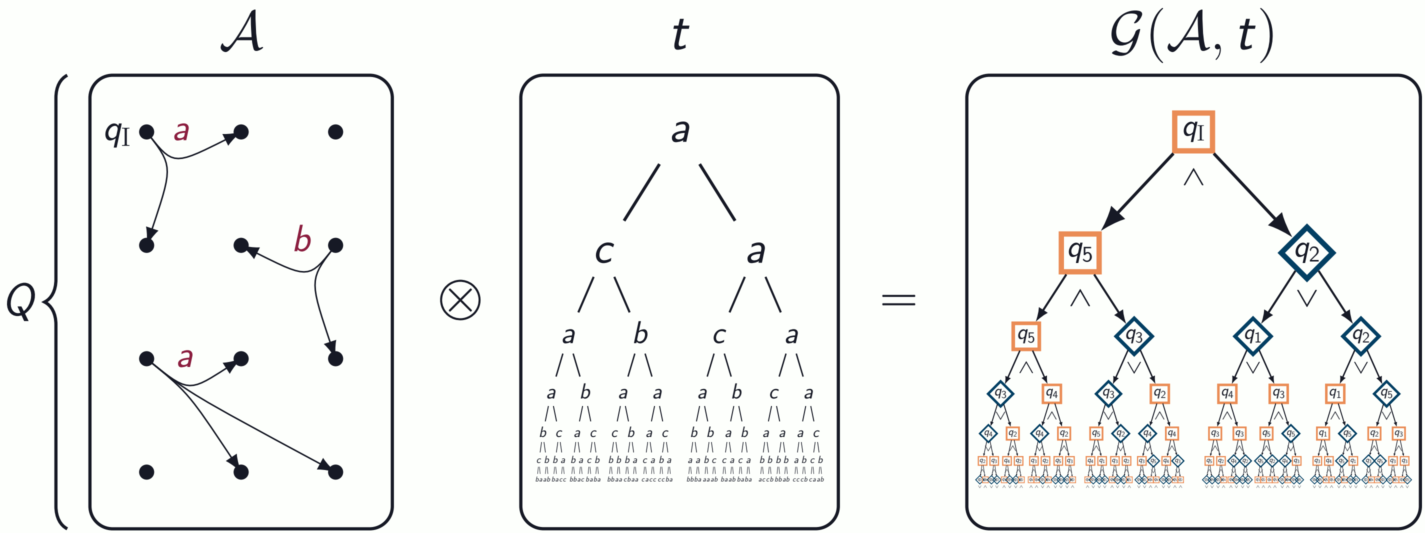

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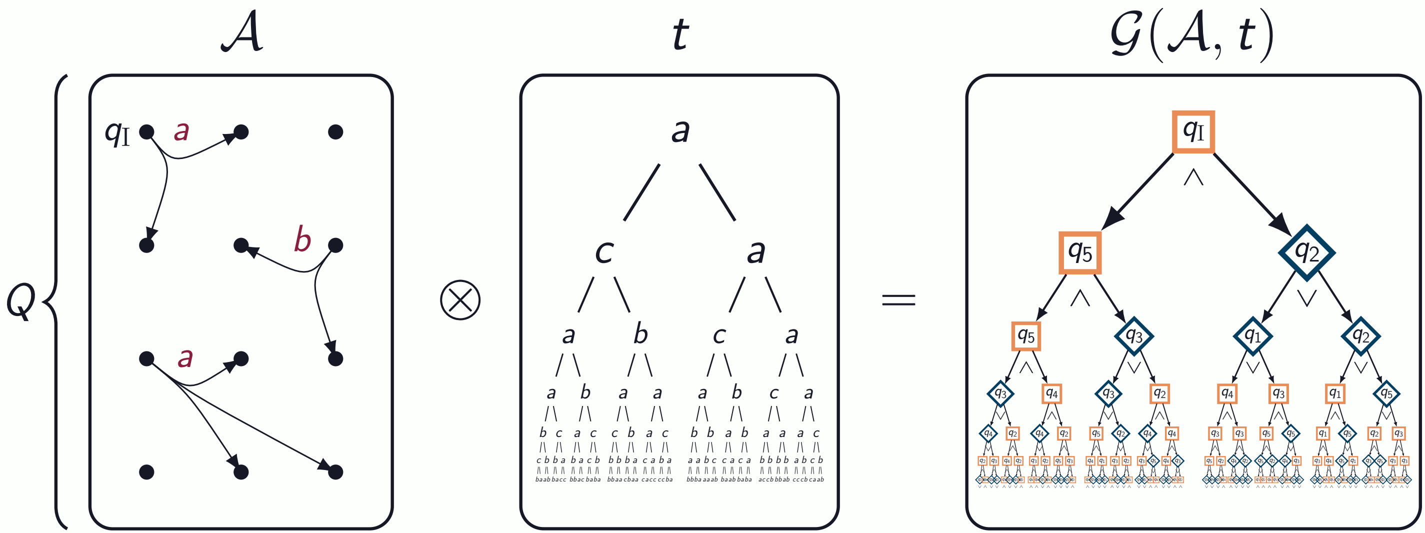

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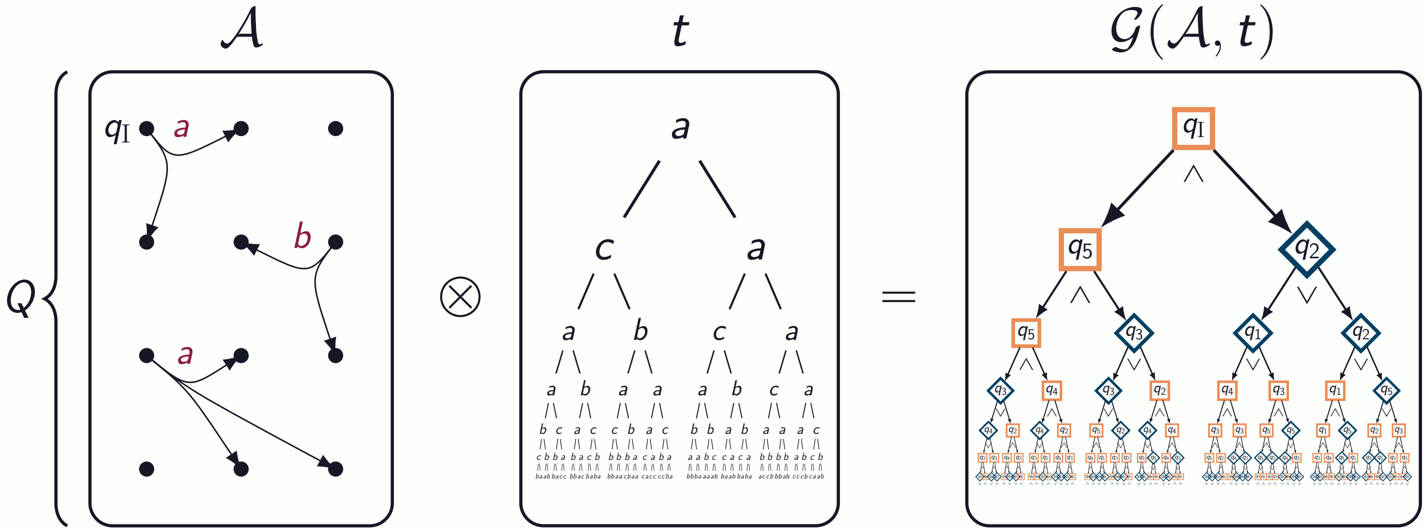

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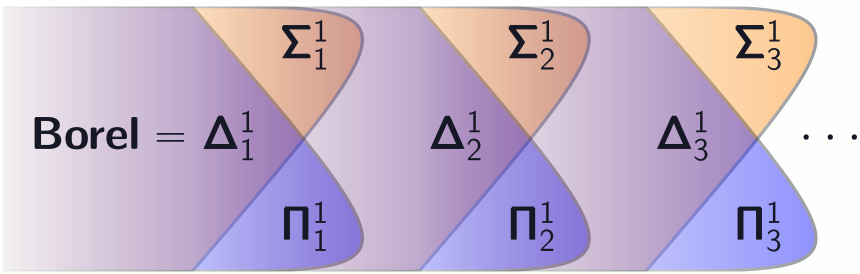

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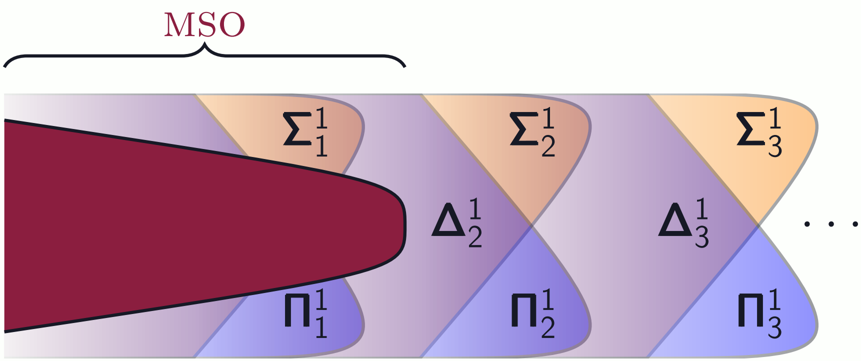

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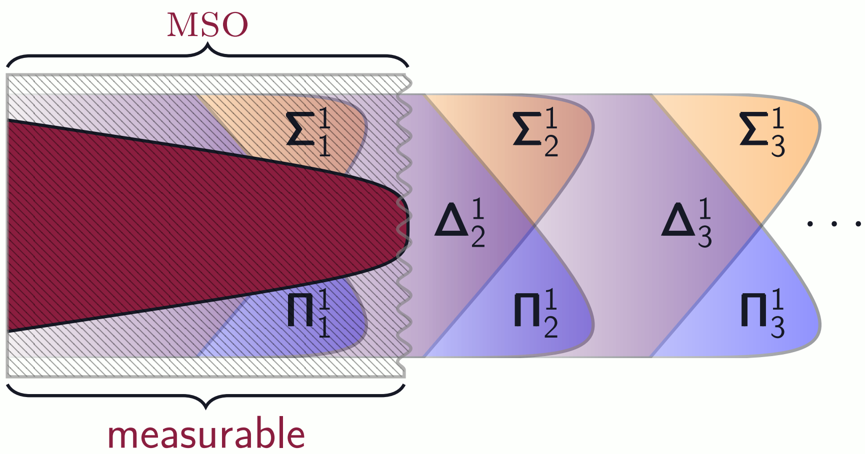

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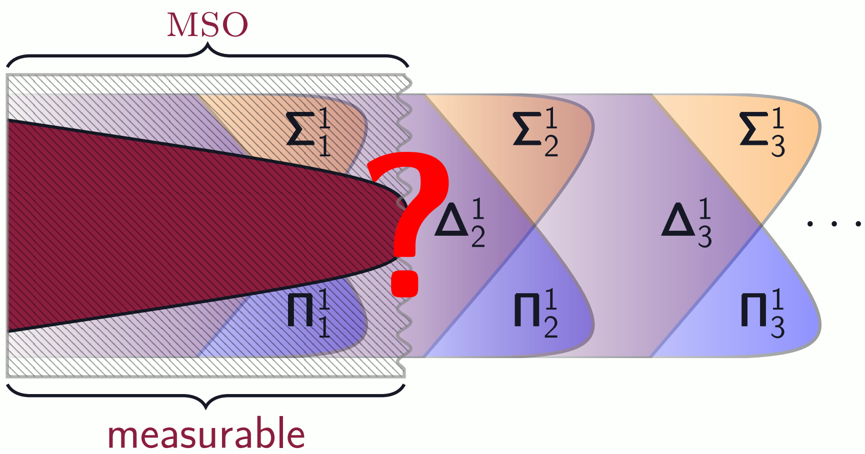

Languages $W_{i,j}$

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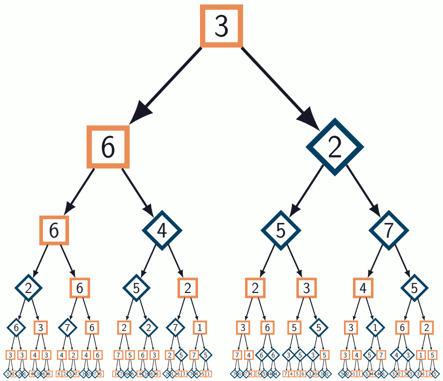

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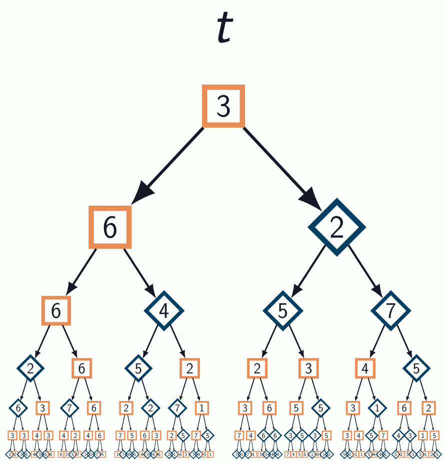

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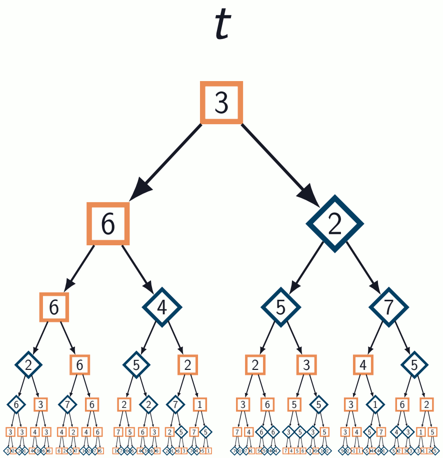

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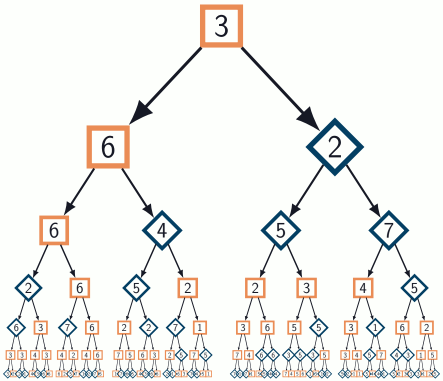

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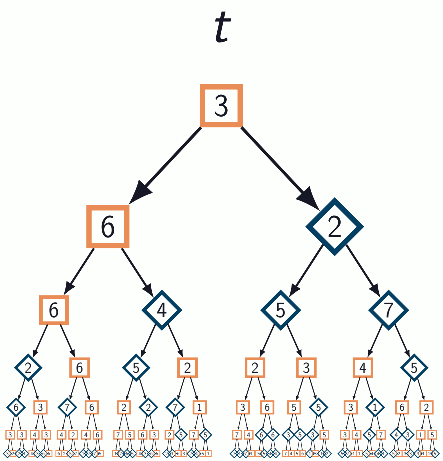

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$$W_{0,1} <_{\mathrm{W}} W_{0,2} <_{\mathrm{W}} W_{0,3} <_{\mathrm{W}} \cdots$$

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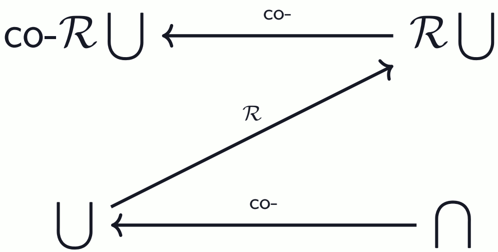

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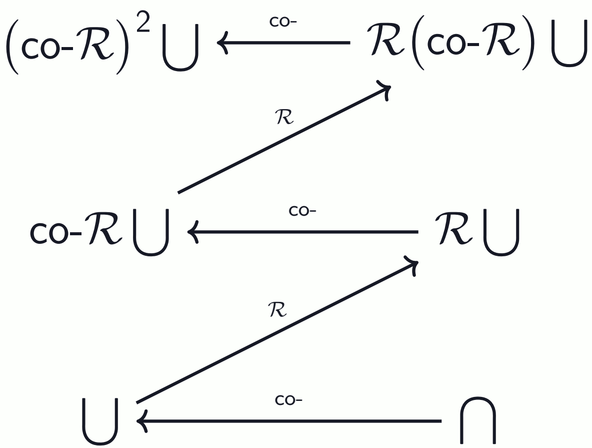

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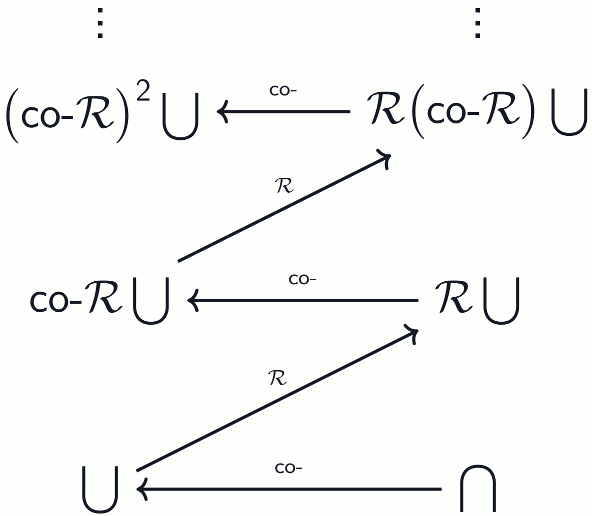

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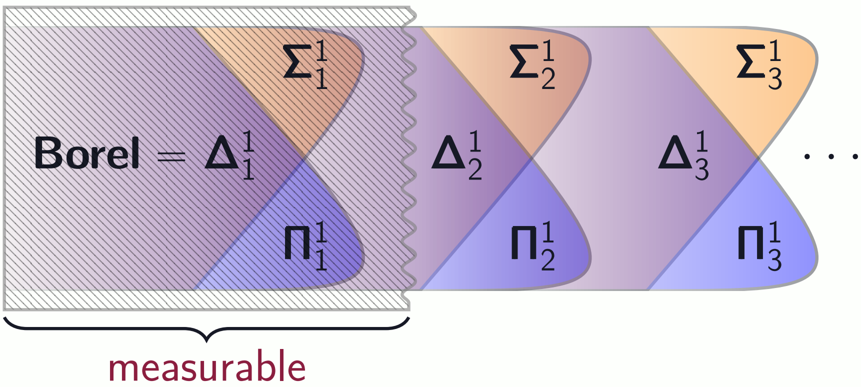

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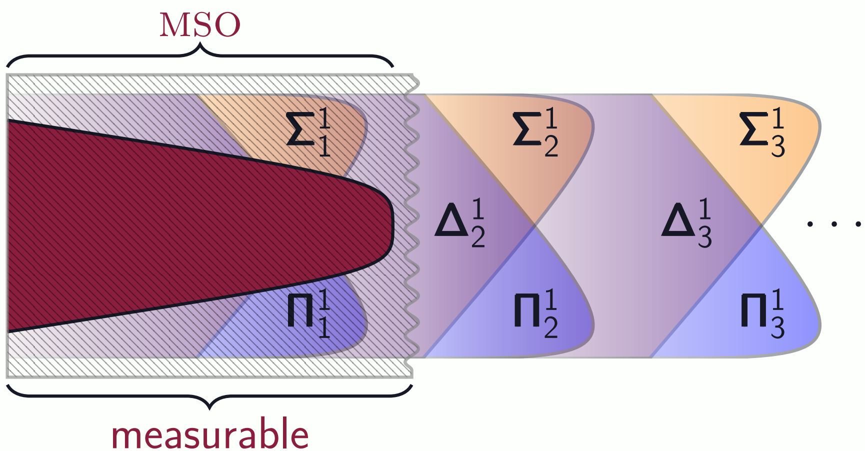

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Theorem (Przybyłko [PhD thesis 2019])

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$$\varphi ::= \exists x.\ \varphi \mid \varphi \vee \varphi \mid \neg\varphi \mid x = y \mid \exists^{\mathrm{fin}} X.\ \varphi \mid x \in X \mid x \preceq y \mid x \leqslant_{\mathrm{lex}} y \mid a(x)$$

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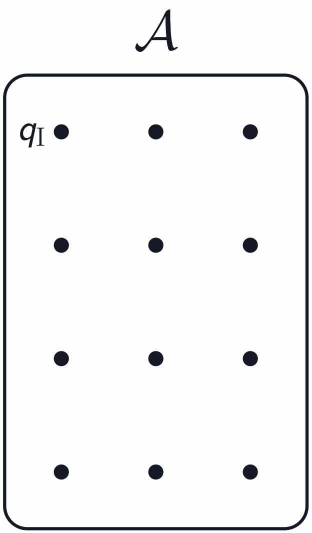

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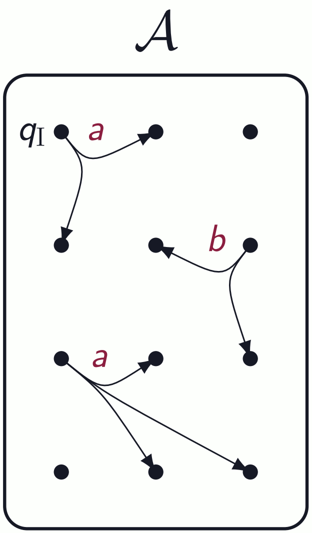

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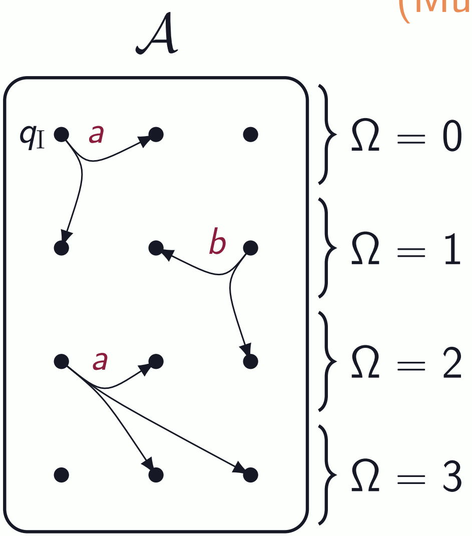

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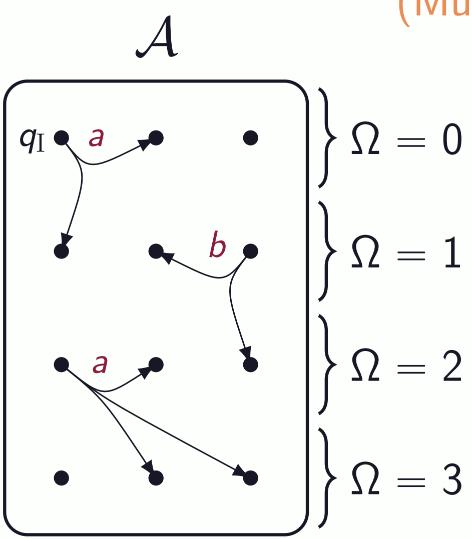

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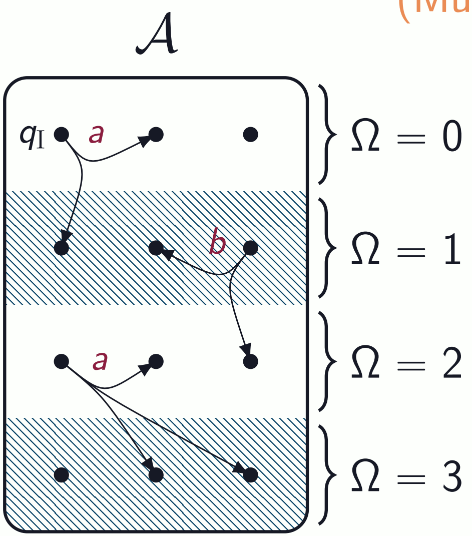

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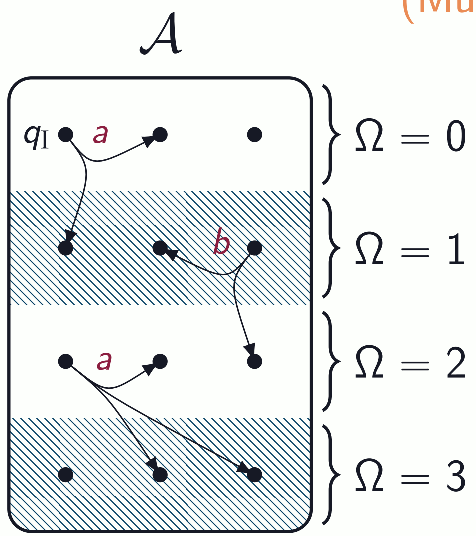

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◇ wins $\mathcal{G}(\mathcal{A}, t)$ **iff** finitely many 

Basic case: safety automata

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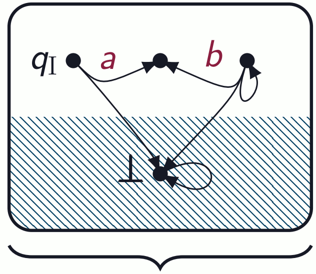
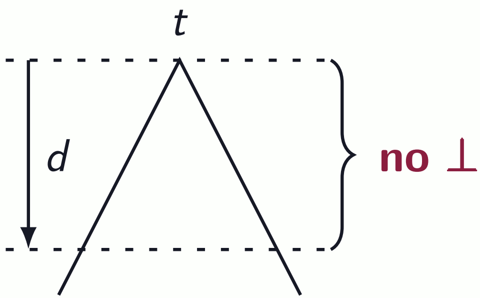
$\mathcal{A}$

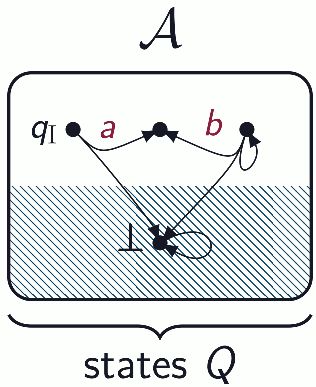


states Q

Basic case: safety automata

$$\mathrm{L}\big(\mathcal{A}^{\geqslant d}\big) \stackrel{\text{def}}{=} \Big\{ t \in \mathrm{Tr}_A \;\Big|\; \Diamond \text{ has a } \textbf{strategy} \text{ in } \mathcal{G}(\mathcal{A}, t) \text{ such that} \Big\}$$





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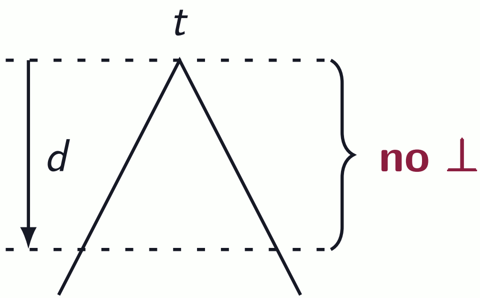

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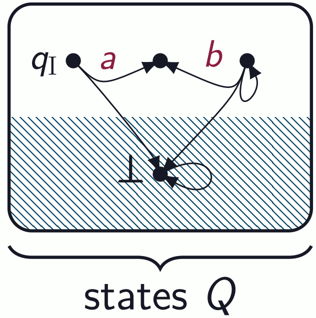

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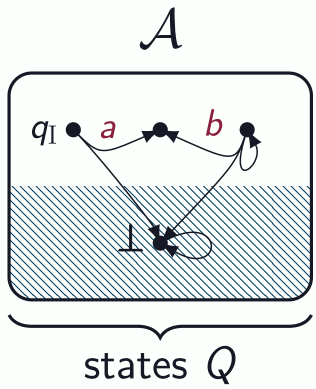

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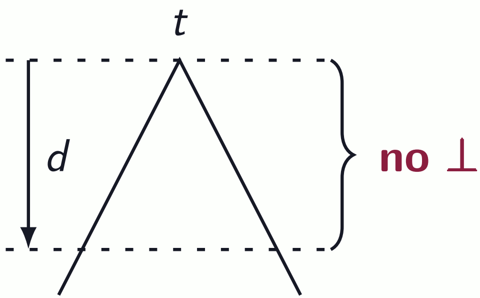

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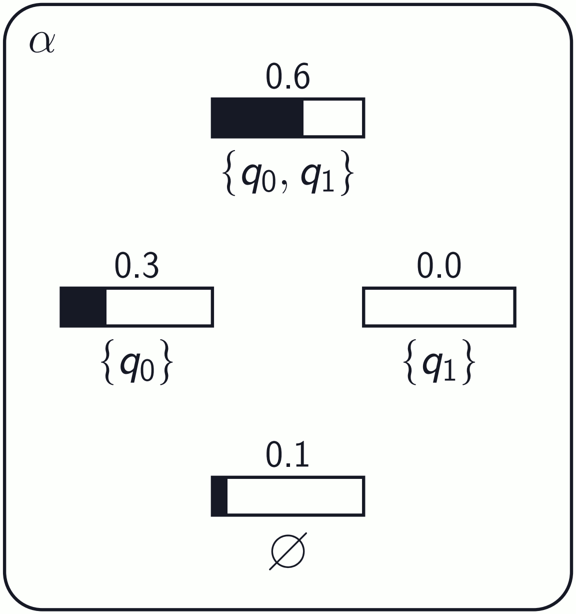

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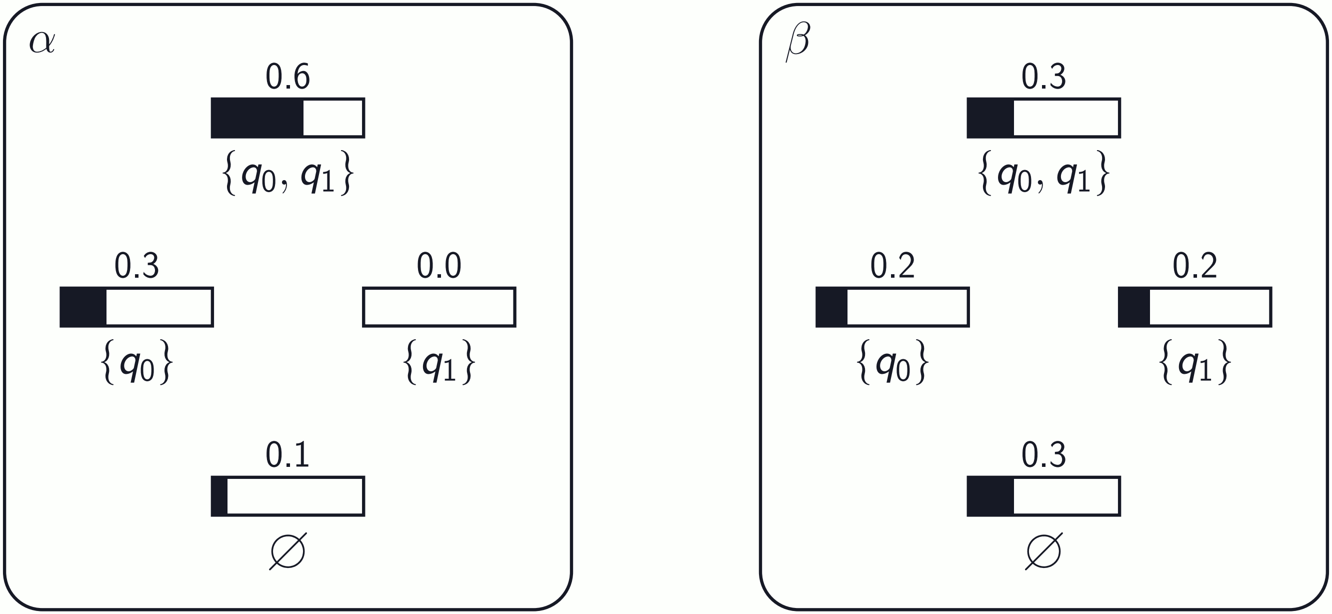

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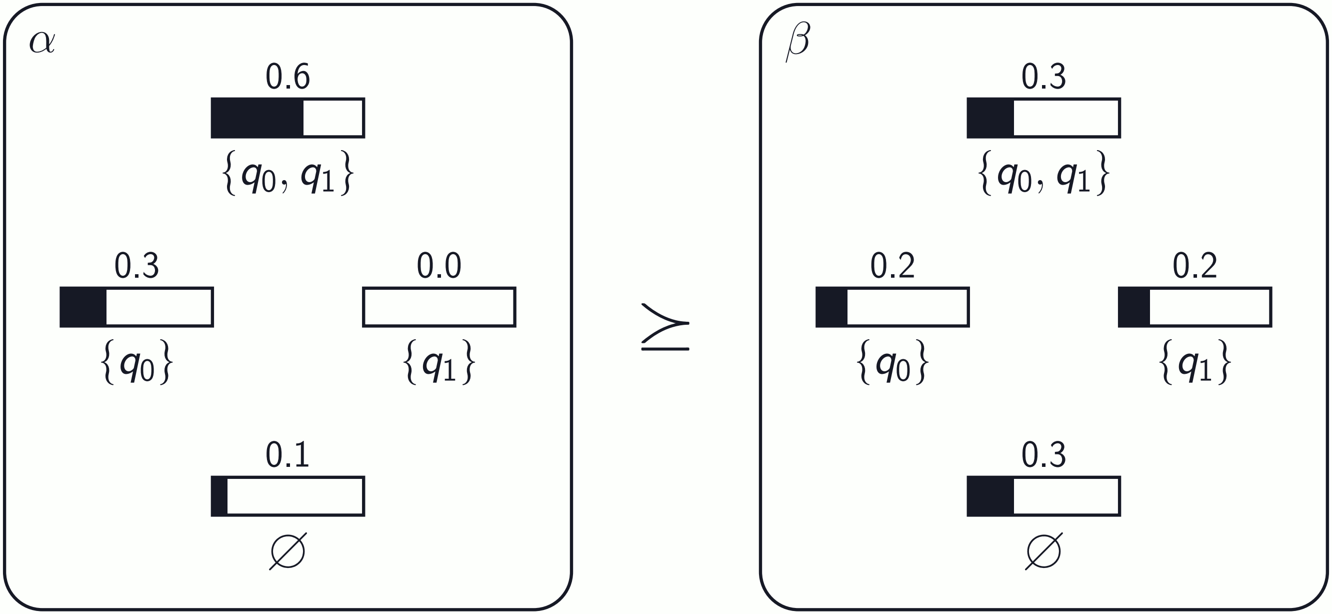

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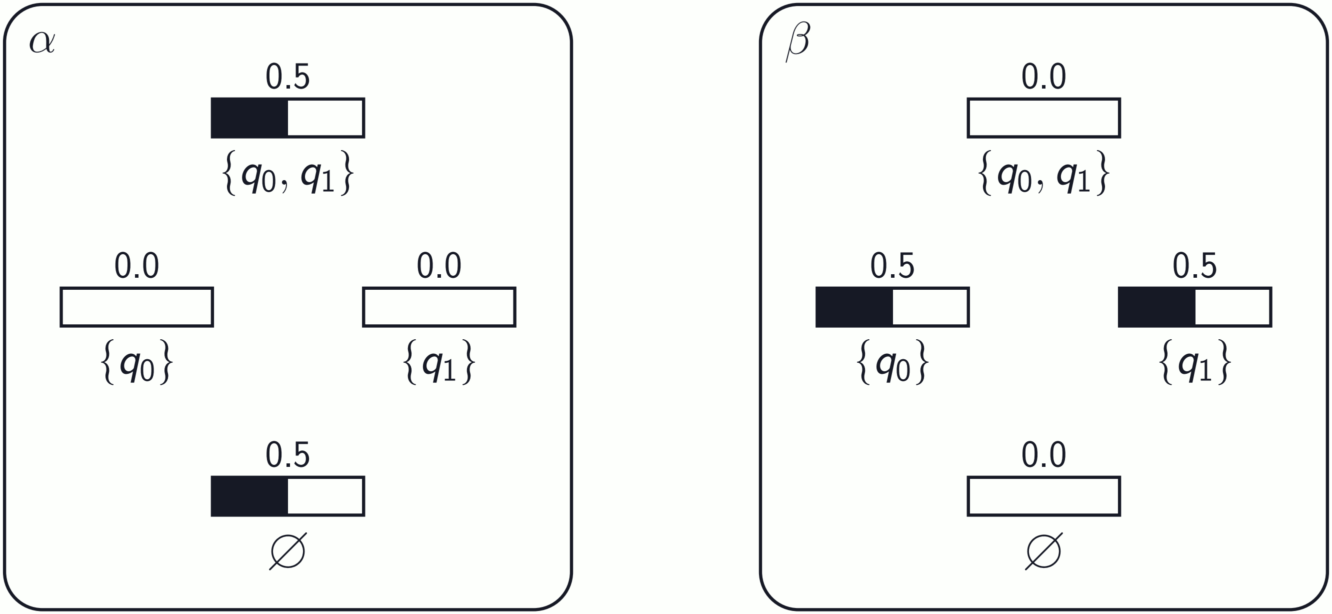

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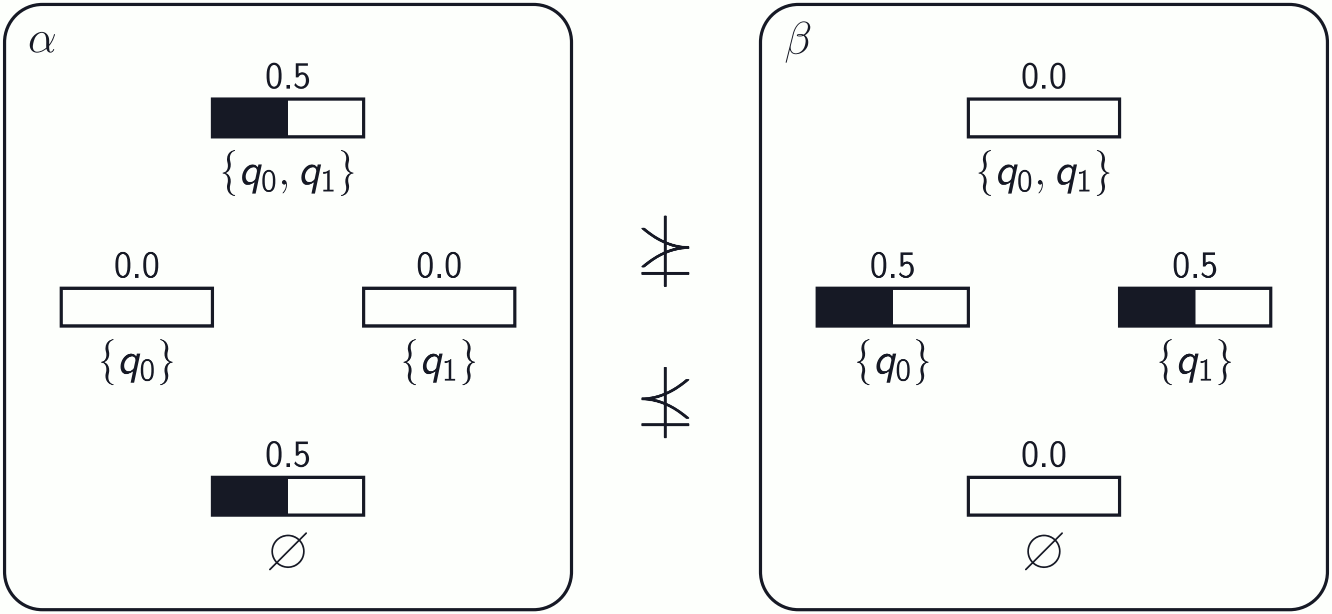

Order on $\mathcal{D}\big(\mathsf{P}(Q)\big)$

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$$\alpha \succeq \beta$$

iff

$$\forall\, \mathcal{U} \subseteq \mathsf{P}(Q),\ \mathcal{U} \text{ upward-closed.}\ \sum_{R \in \mathcal{U}} \alpha(R) \geqslant \sum_{R \in \mathcal{U}} \beta(R)$$

Order on $\mathcal{D}\big(\mathsf{P}(Q)\big)$

$$\alpha \geq \beta$$

"**probabilistic powerdomains**"

(Saheb-Djahromi [1980])

(Jones, Plotkin [1989])

iff

$$\forall\, \mathcal{U} \subseteq \mathsf{P}(Q),\ \mathcal{U} \text{ upward-closed. } \sum_{R\in\mathcal{U}} \alpha(R) \geqslant \sum_{R\in\mathcal{U}} \beta(R)$$

Order on $\mathcal{D}(\mathrm{P}(Q))$

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iff

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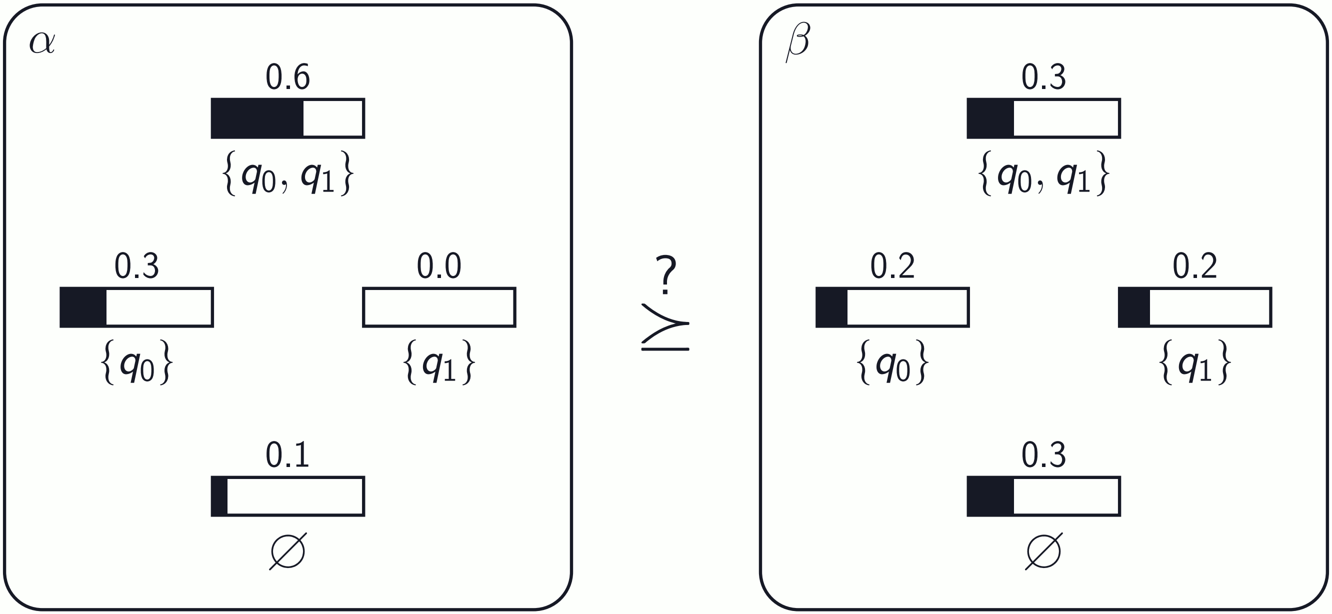

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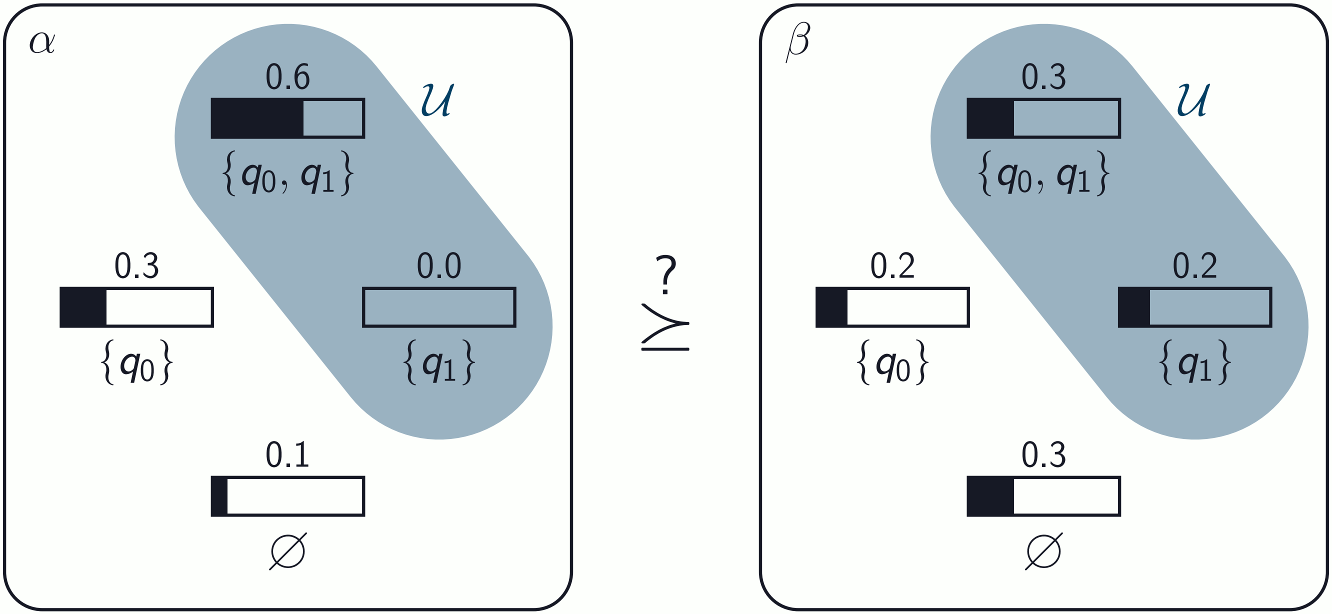

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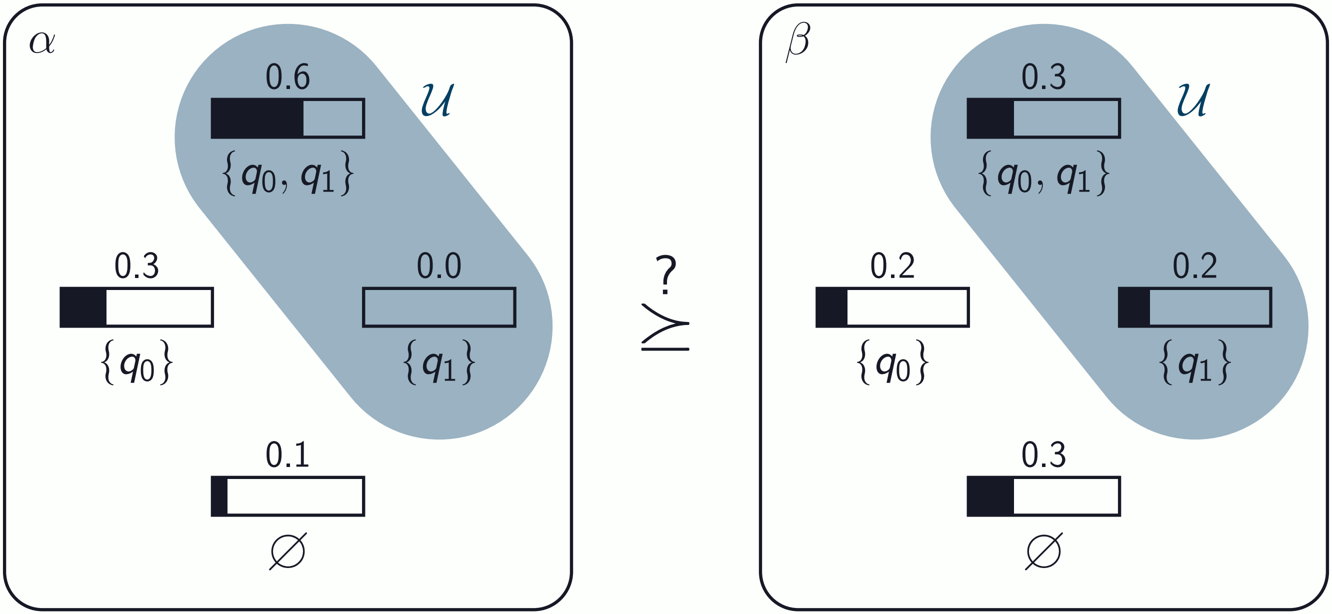

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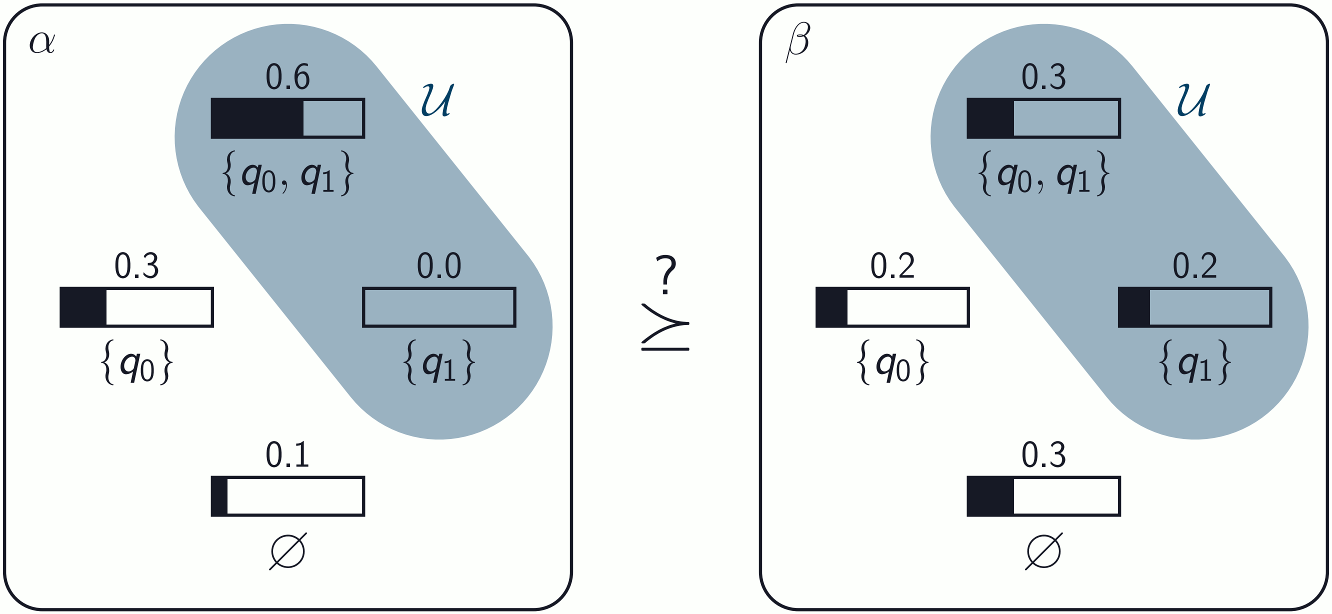

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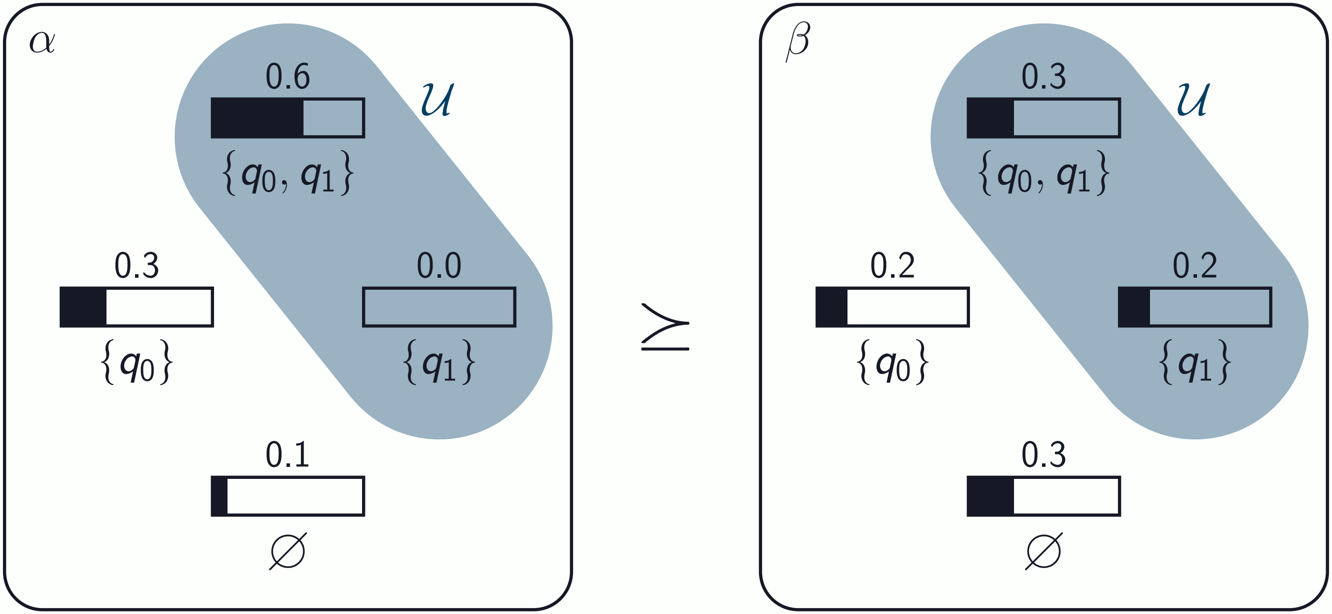

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[in $\mathbb{R}^{\mathrm{P}(Q)}$]

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using **First-order** theory of reals (Tarski [1951])

General case of weak-mso

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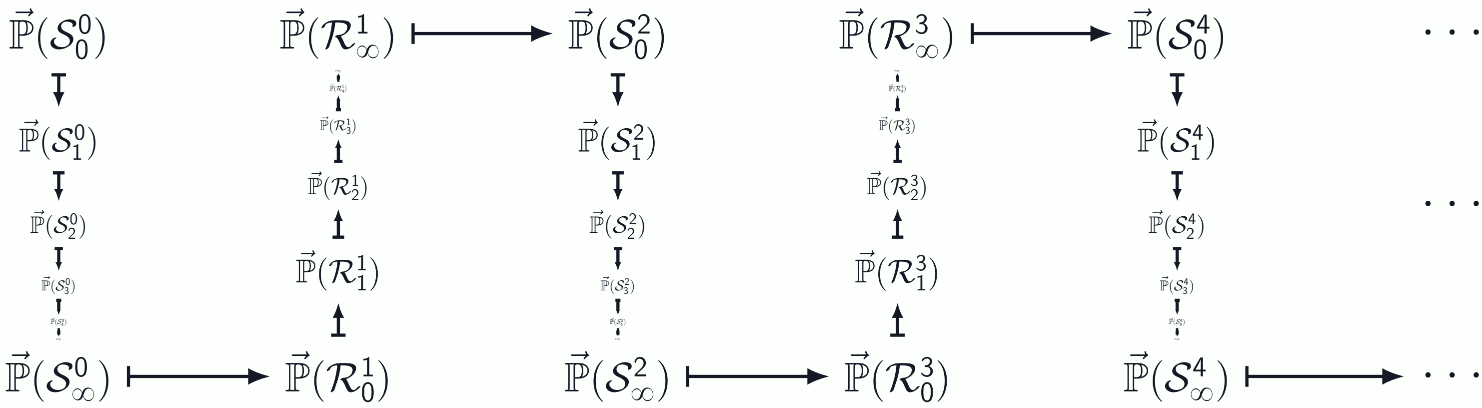

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If $(\alpha_\eta)_{\eta<\Theta}$ is a chain $\left(\alpha_\eta \prec \alpha_\theta \text{ for } \eta < \theta\right)$,

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Conjecture

$$\mathbb{P}\left(\mathrm{L}(\mathcal{A})\right) = \mathbb{P}\left(\mathrm{L}(\mathcal{A}^{\leqslant\omega})\right)$$

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