Computing measures of sets of infinite trees definable in weak-MSO logic

Damian Niwiński, Marcin Przybyłko, Michał Skrzypczak

University of Warsaw

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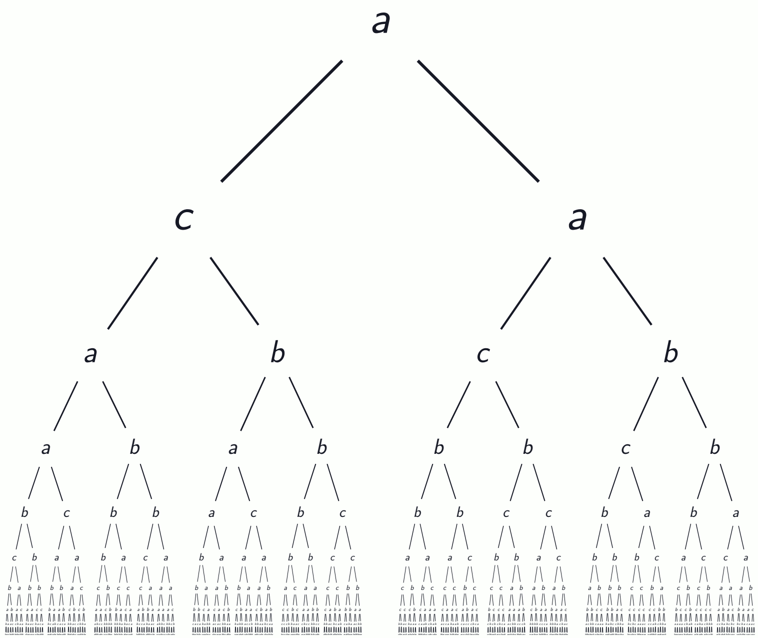

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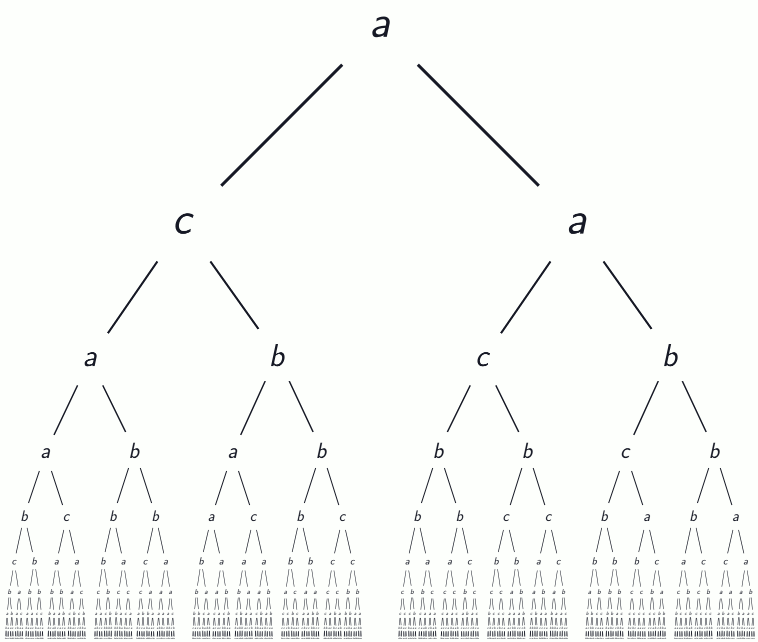

Regular languages

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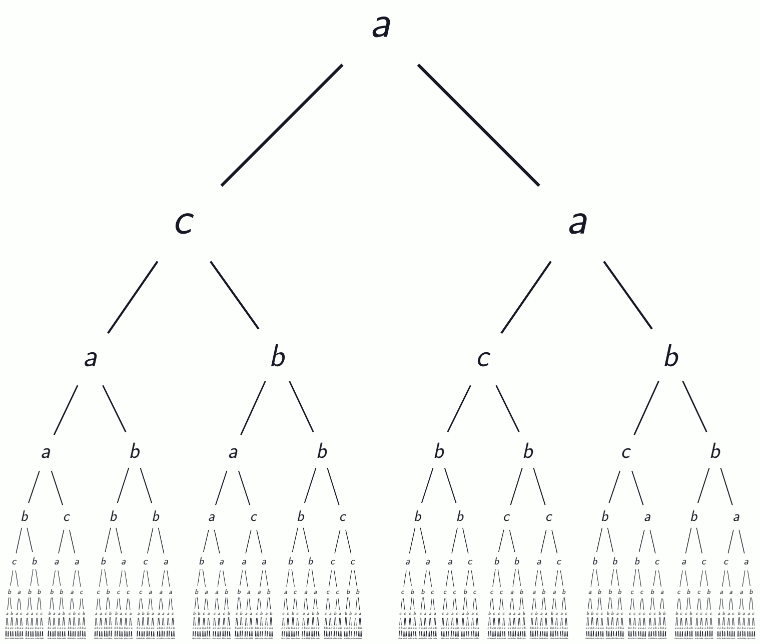

Regular languages

Monadic Second-order logic (MSO):

$$\varphi ::= \exists X.\ \varphi \mid \exists x.\ \varphi \mid \varphi \vee \varphi \mid \neg\varphi$$

$$x \in X \mid x \preceq y \mid x \leqslant_{\mathrm{lex}} y \mid a(x)$$

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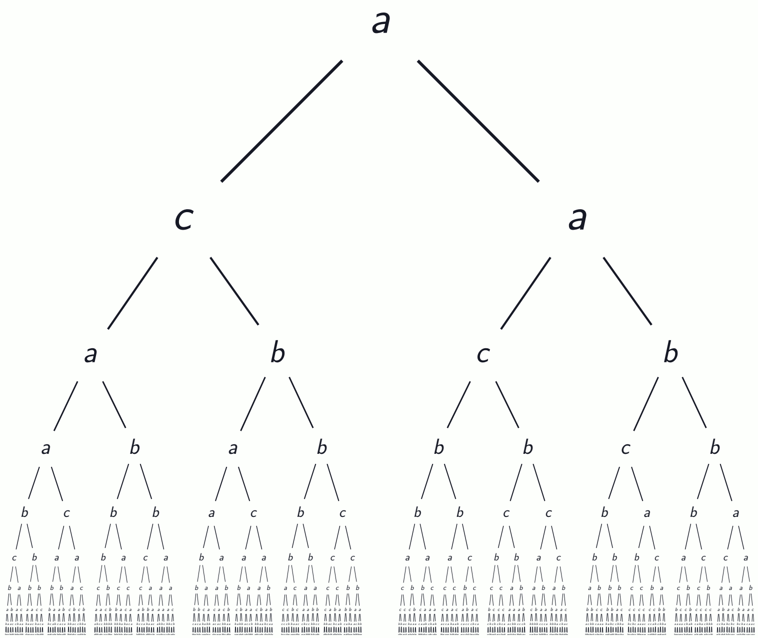

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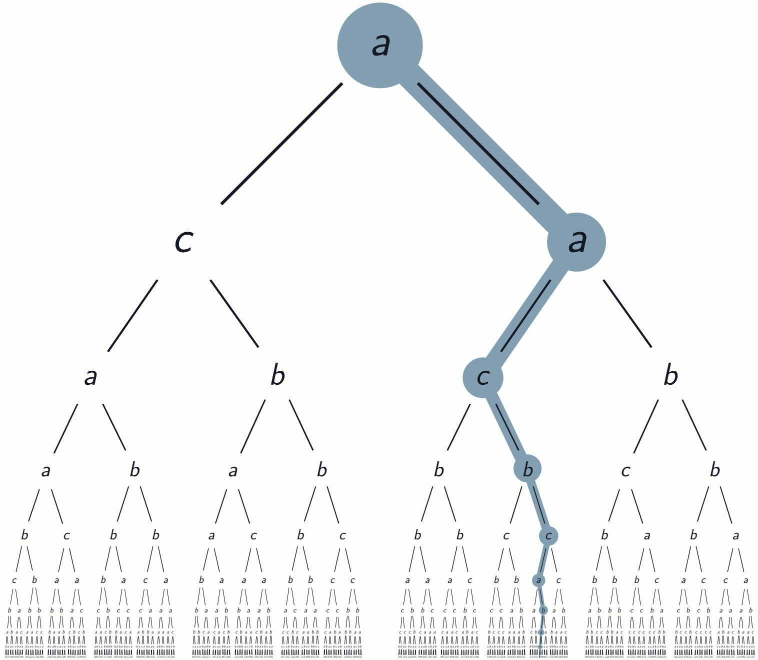

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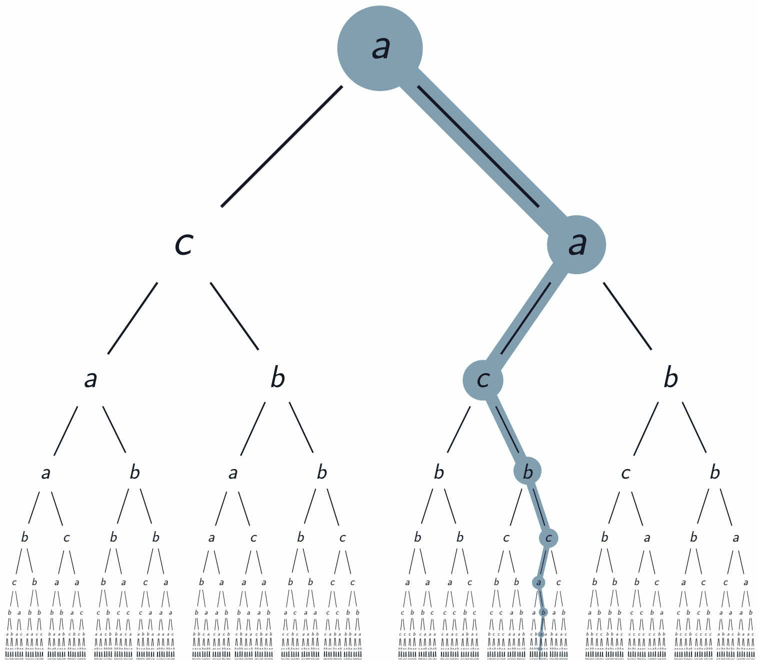

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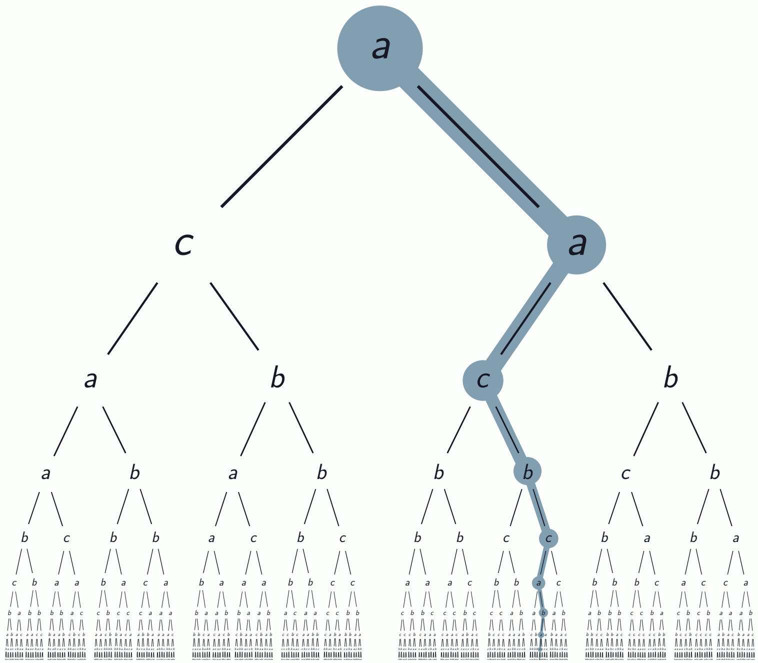

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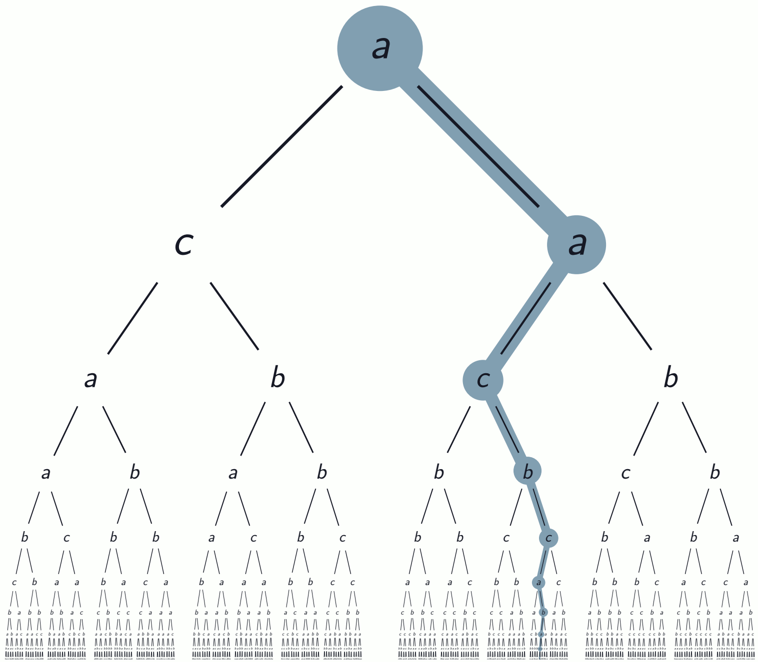

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Logics' zoo

(on infinite trees)

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weak-MSO $\subsetneq$ MSO

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alternation-free μ-calculus $\supsetneq$ CTL

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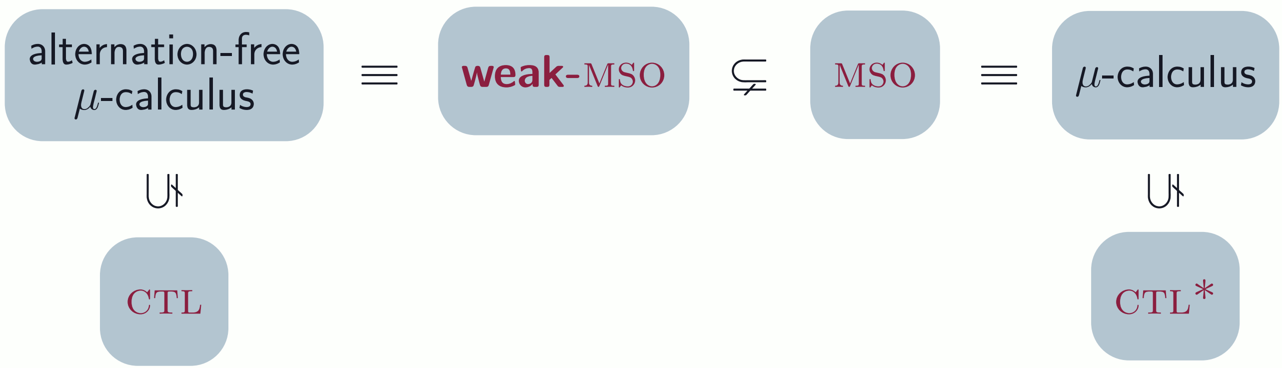

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alternation-free μ-calculus $\equiv$ **weak-**MSO $\subsetneq$ MSO $\equiv$ μ-calculus

alternation-free μ-calculus	weak-MSO	MSO	μ-calculus
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CTL	FO		CTL^*

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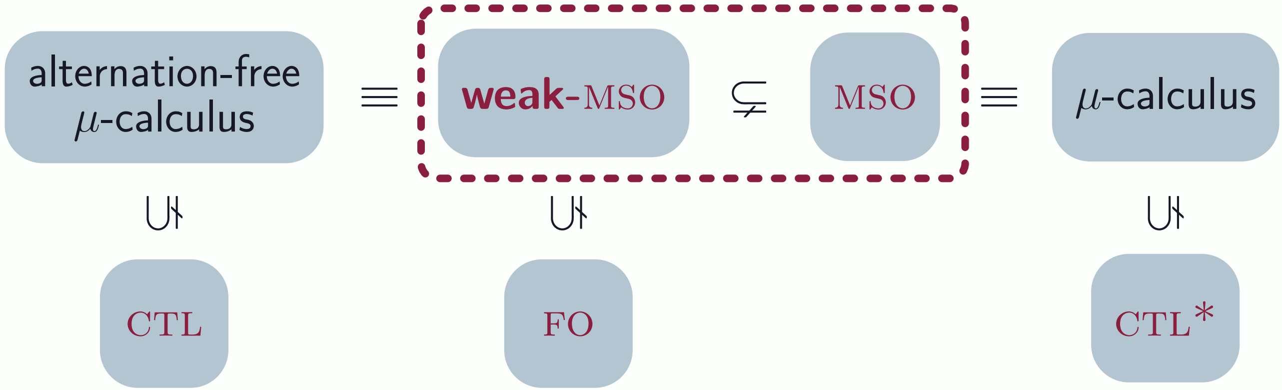

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Theorem (Gogacz, Michalewski, Mio, S. [2017])

Every regular language of infinite trees is **measurable**.

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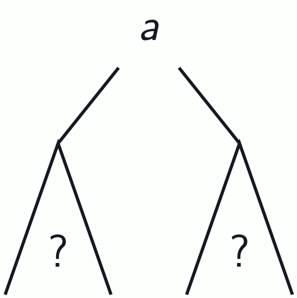

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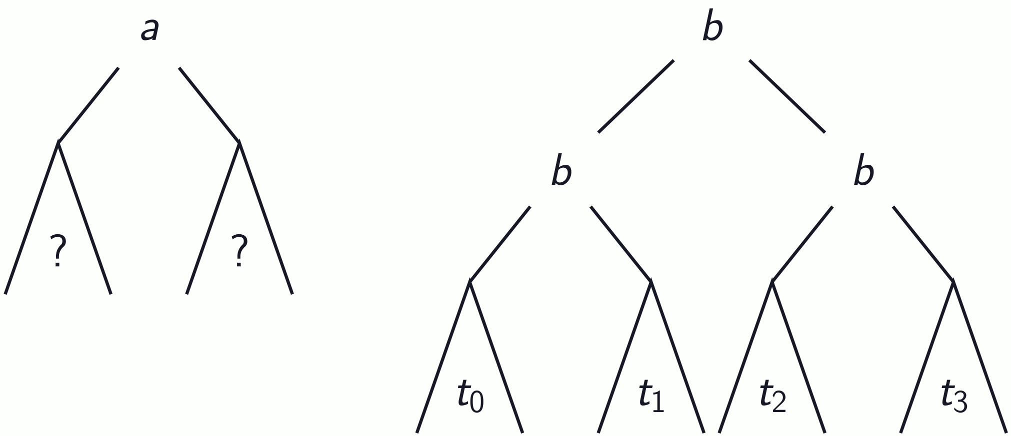

$t_0, t_1, t_2, t_3 \in \mathrm{L}(\varphi)$

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What if $x_0, x_1 \in [0, 1]$ **?**

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CTL $\subseteq$ weak-MSO $\rightsquigarrow$ model checking of **stochastic branching processes**...

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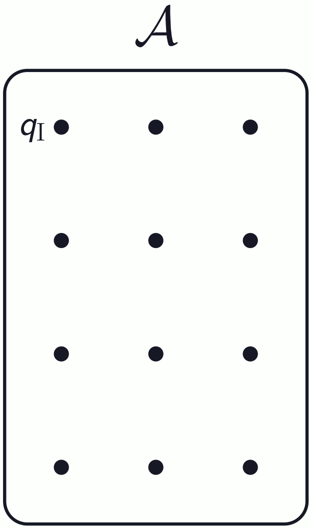

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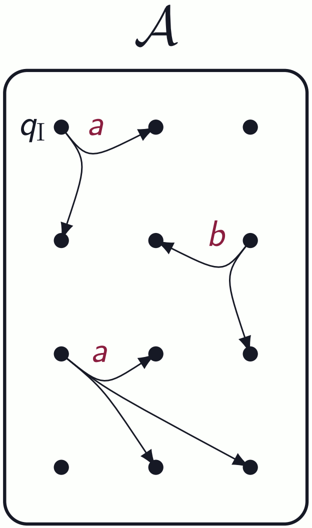

$$\delta(q_I, a) = (q_5, L) \wedge (q_2, L)$$
$$\delta(q_I, b) = (q_4, R) \vee \big((q_1, L) \wedge (q_1, R)\big)$$
$$\delta(q_I, c) = (q_1, R) \vee (q_3, R)$$
$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

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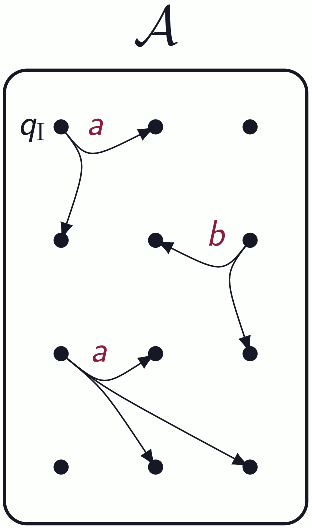
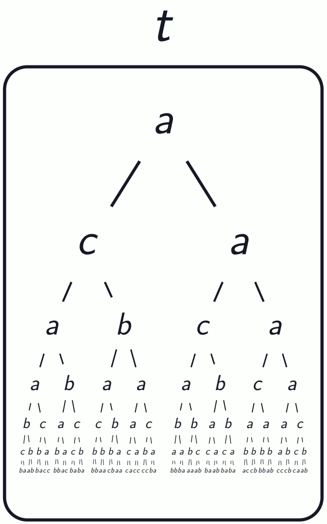


$\bigotimes$



$$\delta(q_{\mathrm{I}}, a) = (q_5, \mathrm{L}) \wedge (q_2, \mathrm{L})$$
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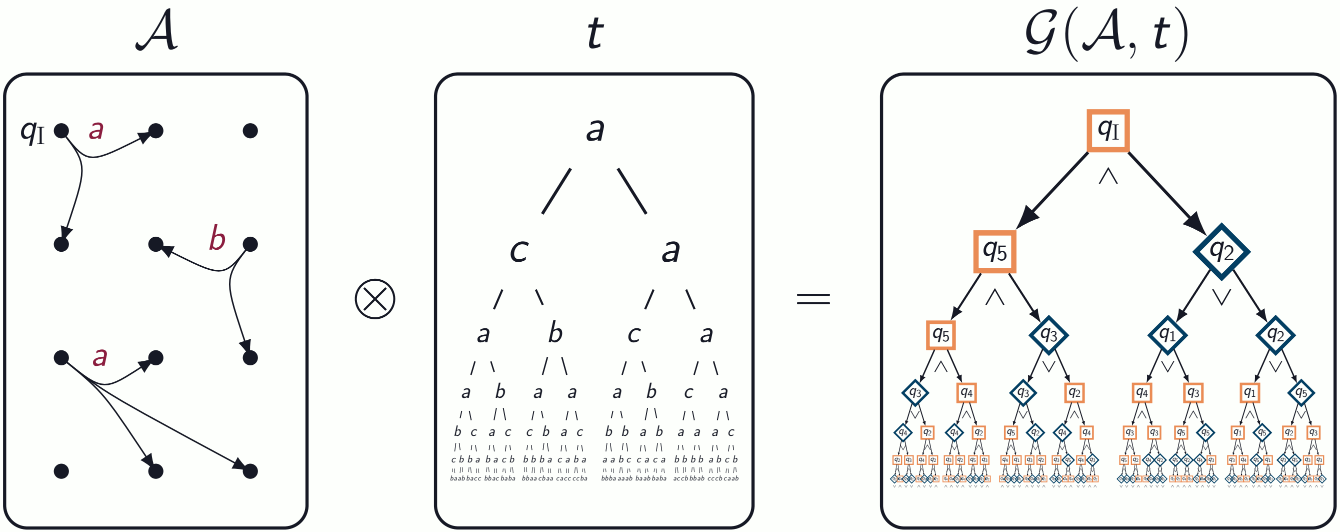

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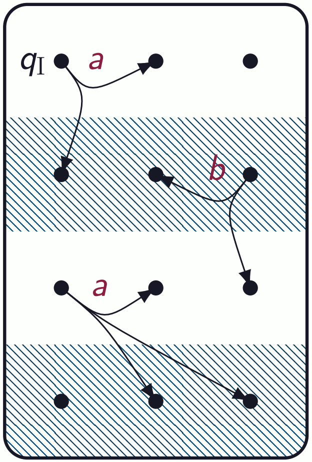
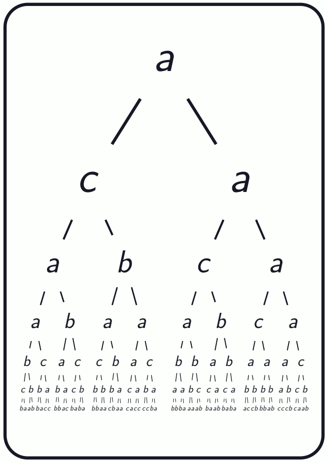


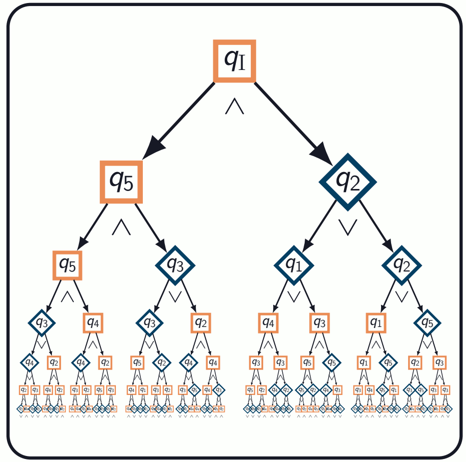
$\otimes$

t



$=$

$\mathcal{G}(\mathcal{A}, t)$



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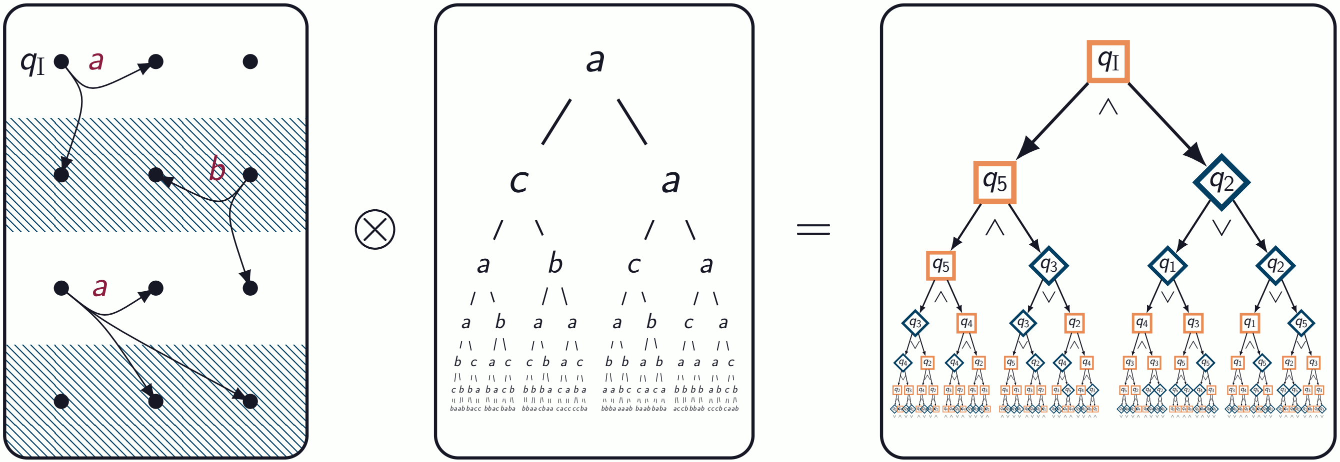

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Game played between ◇ and □

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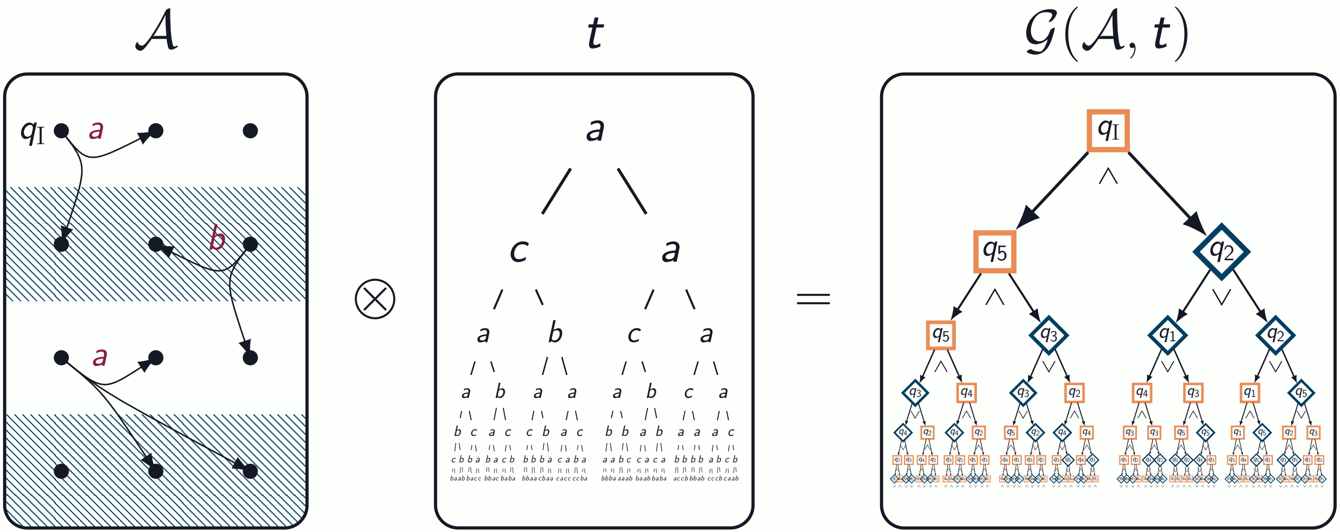

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⋮ ⋮ ⋮ ⋮

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◇ wins **iff** finitely many ●

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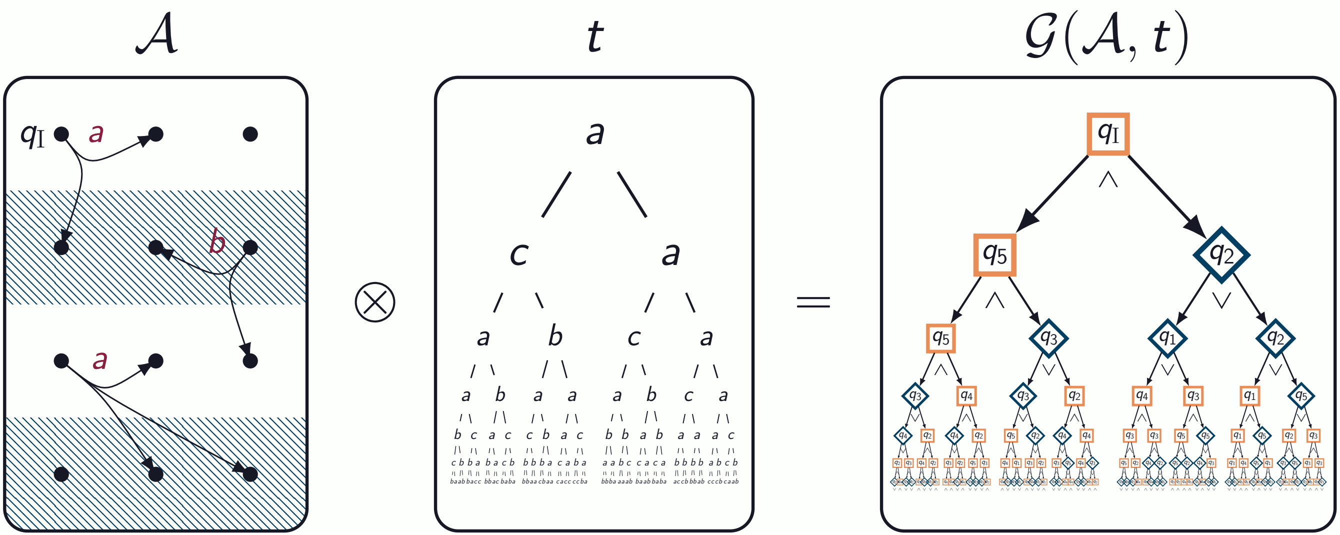

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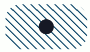
◇ wins **iff** finitely many 

$$\mathrm{L}(\mathcal{A}) = \left\{ t \in \mathrm{Tr}_A \mid \diamond \text{ has a winning strategy in } \mathcal{G}(\mathcal{A}, t) \right\}.$$

Basic case: safety automata

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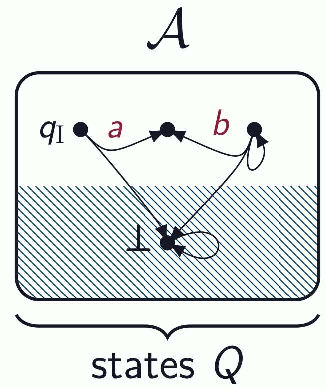
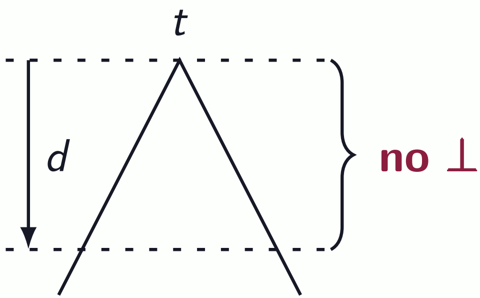
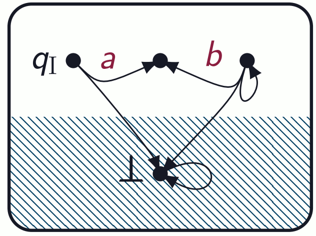


Basic case: **safety automata**

$$\mathrm{L}\big(\mathcal{A}^{\geqslant d}\big) \overset{\text{def}}{=} \Big\{ t \in \mathrm{Tr}_A \;\Big|\; \Diamond \text{ has a } \textbf{strategy} \text{ in } \mathcal{G}(\mathcal{A}, t) \text{ such that} \Big\}$$

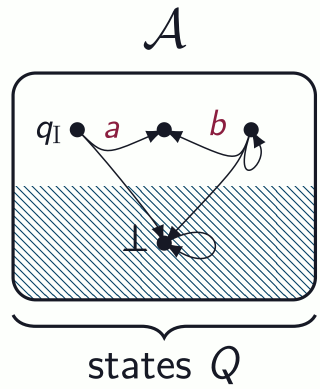


$\mathcal{A}$

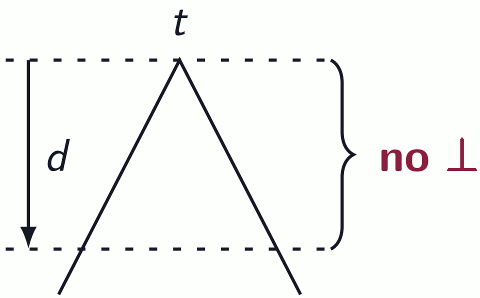


states Q

Basic case: safety automata



$L(\mathcal{A}^{\geqslant d}) \stackrel{\text{def}}{=} \{ t \in \mathrm{Tr}_A \mid$ ◇ has a **strategy** in $\mathcal{G}(\mathcal{A}, t)$ such that



$\}$

$$L(\mathcal{A}^{\geqslant 0}) \supseteq L(\mathcal{A}^{\geqslant 1}) \supseteq L(\mathcal{A}^{\geqslant 2}) \supseteq \cdots \supseteq L(\mathcal{A})$$

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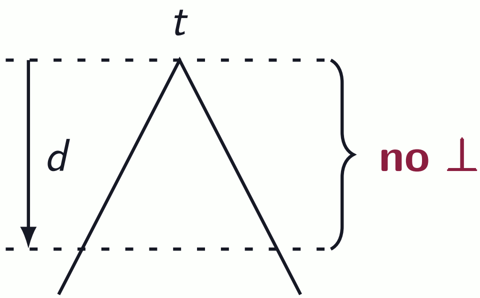

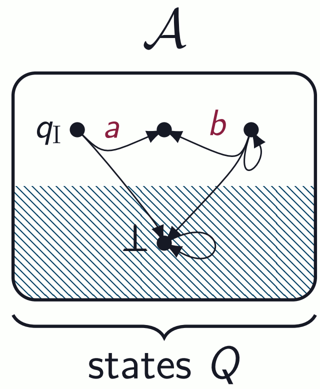

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König's Lemma ⇝

$$\mathrm{L}(\mathcal{A}) = \bigcap_{d \in \mathbb{N}} \mathrm{L}\big(\mathcal{A}^{\geqslant d}\big)$$

From **languages** to **distributions**

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\rightsquigarrow & & \rightsquigarrow & & \rightsquigarrow & & & & \rightsquigarrow \\
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Order on $\mathcal{D}\big(\mathsf{P}(Q)\big)$

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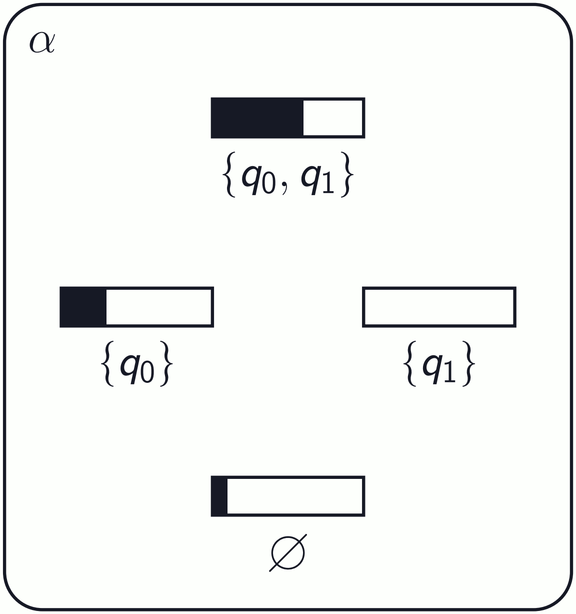

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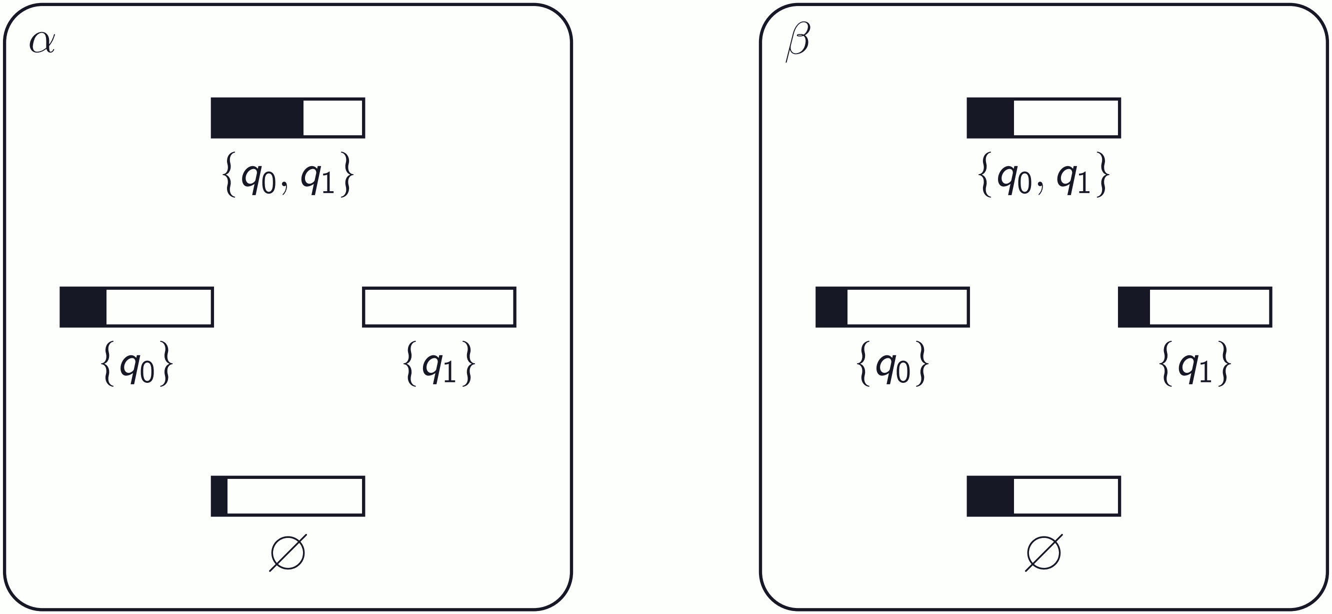

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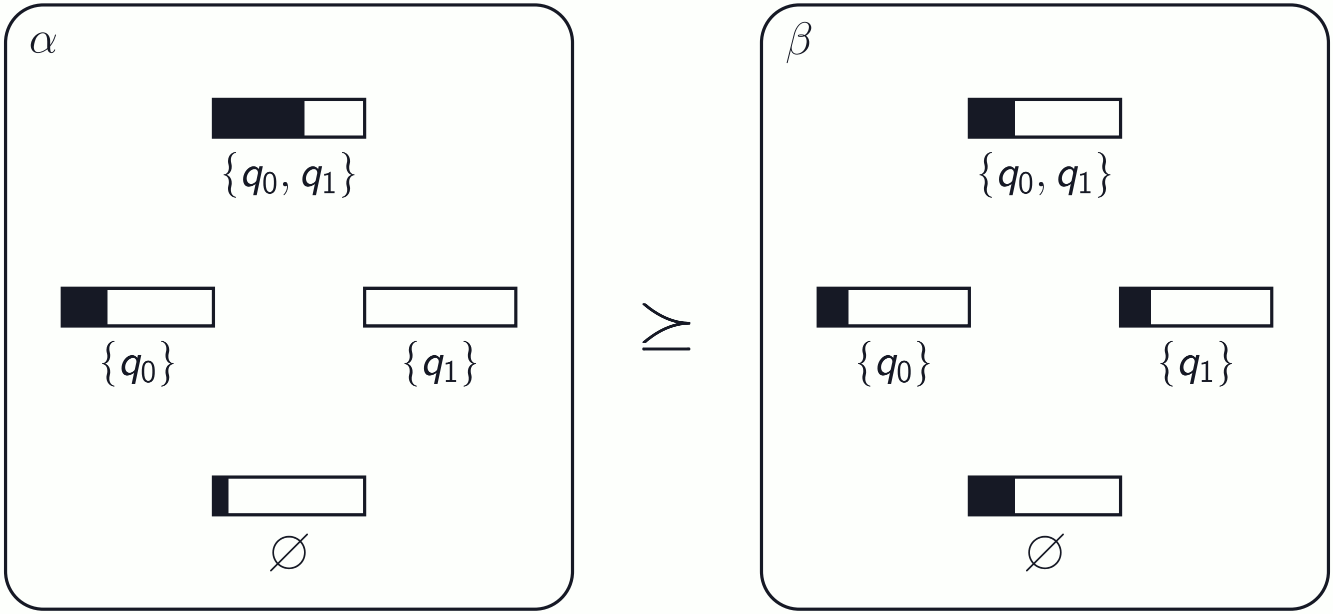

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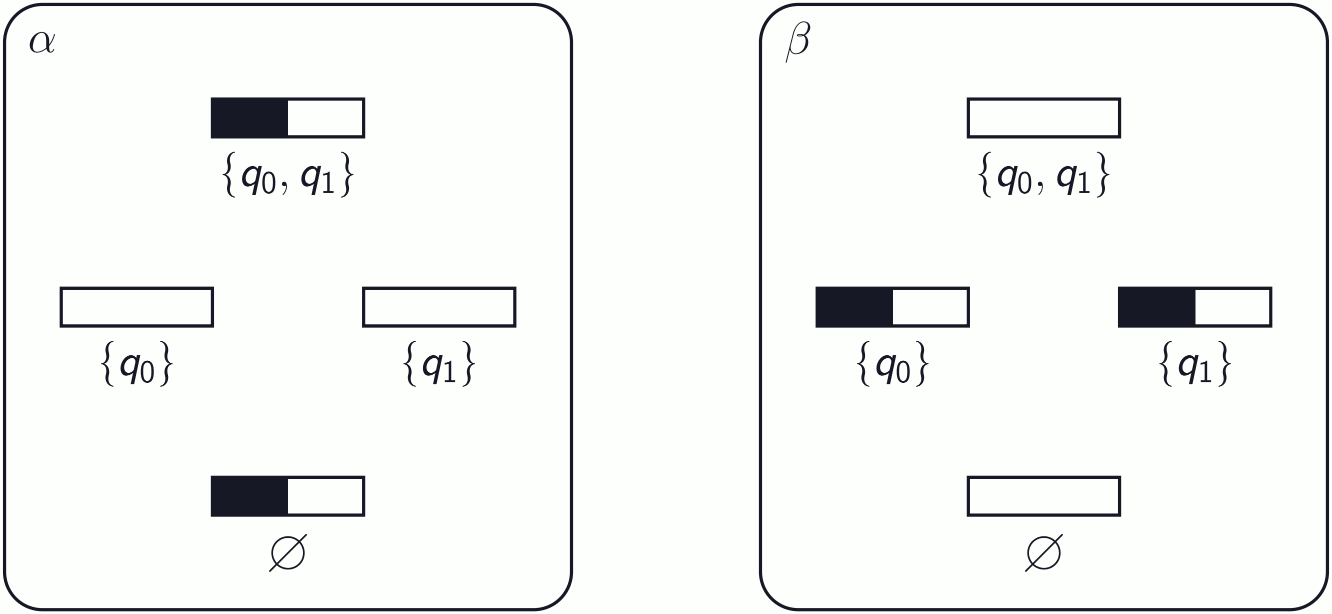

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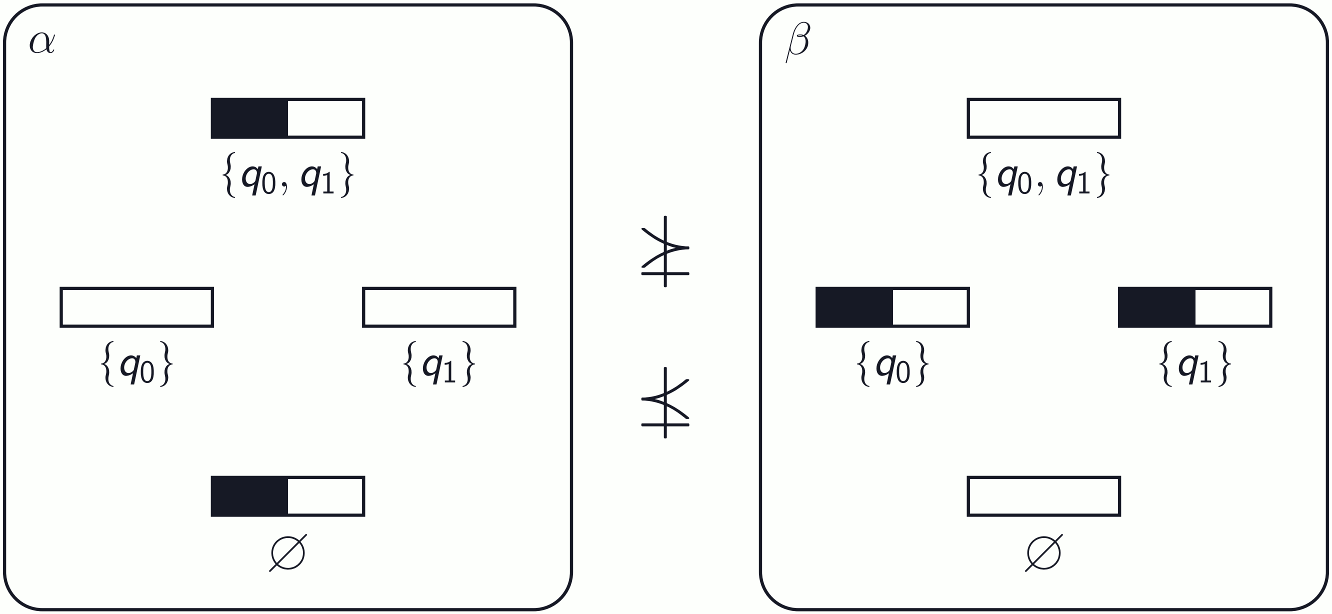

Order on $\mathcal{D}\big(\mathsf{P}(Q)\big)$

Order on $\mathcal{D}\big(\mathsf{P}(Q)\big)$

$$\alpha \succeq \beta$$

iff

$$\forall\, \mathcal{U} \subseteq \mathsf{P}(Q),\ \mathcal{U} \text{ upward-closed.}\ \sum_{R\in\mathcal{U}} \alpha(R) \geqslant \sum_{R\in\mathcal{U}} \beta(R)$$

Order on $\mathcal{D}\big(\mathsf{P}(Q)\big)$

$$\alpha \succeq \beta$$

"**probabilistic powerdomains**"

(Saheb-Djahromi [1980])

(Jones, Plotkin [1989])

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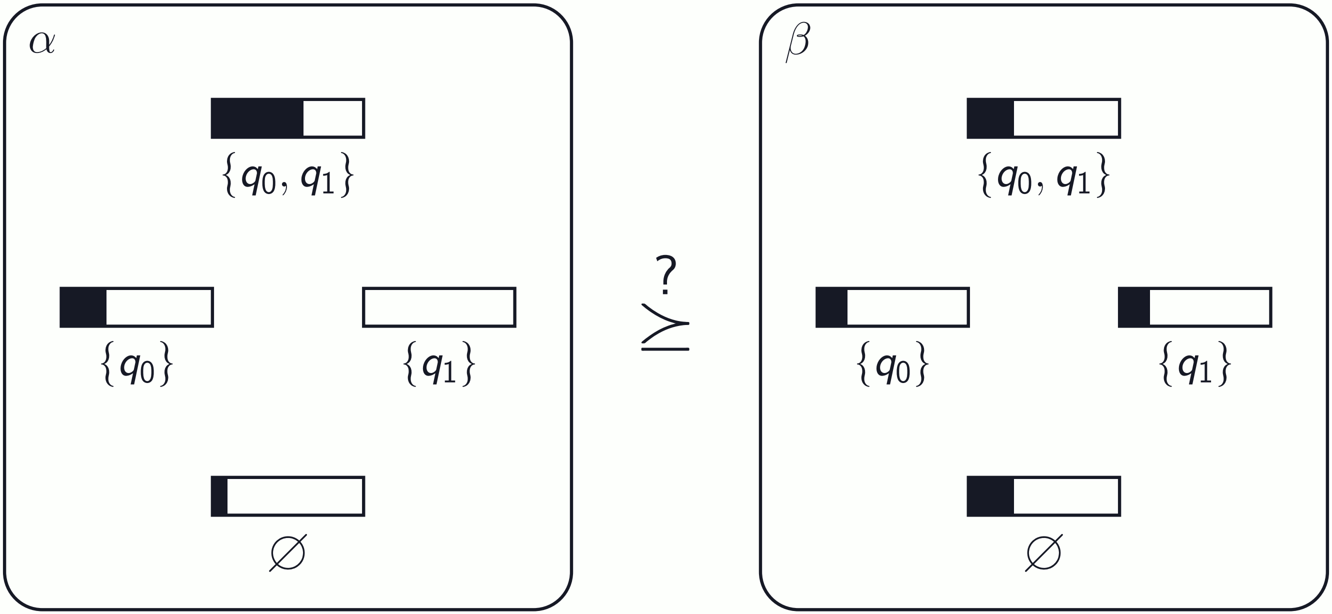

Order on $\mathcal{D}(\mathrm{P}(Q))$

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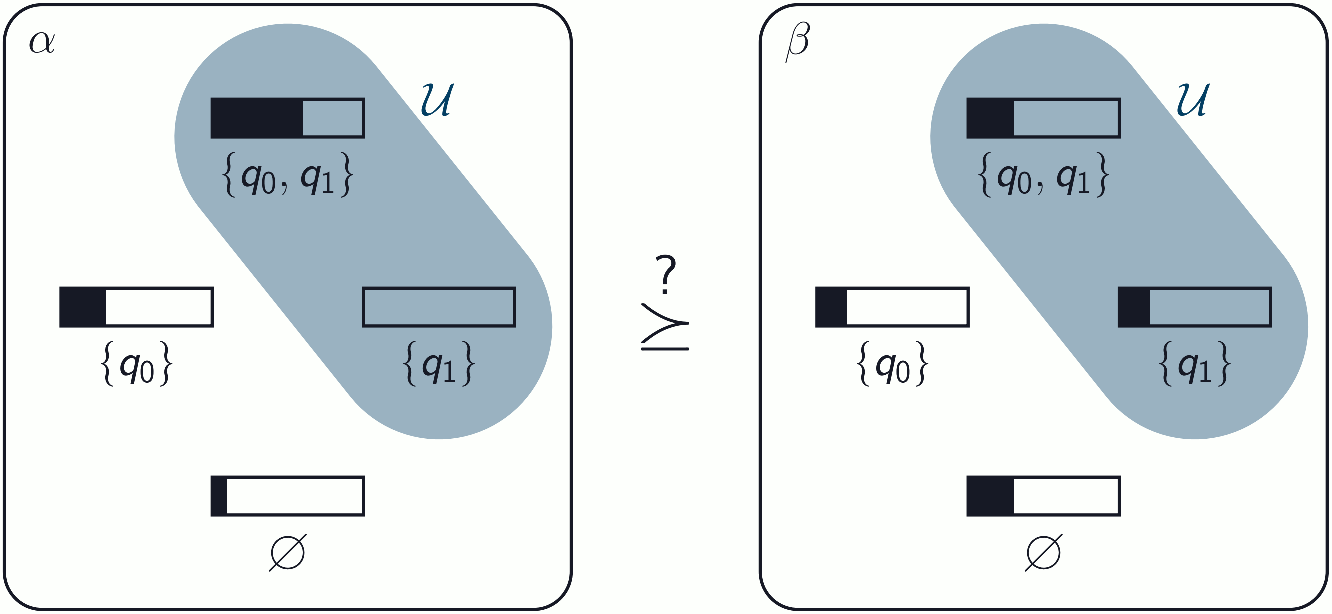

Order on $\mathcal{D}(\mathrm{P}(Q))$

$\alpha \succeq \beta$ **iff** $\forall \mathcal{U} \subseteq \mathrm{P}(Q),\ \mathcal{U}$ upward-closed. $\sum_{R\in\mathcal{U}} \alpha(R) \geqslant \sum_{R\in\mathcal{U}} \beta(R)$

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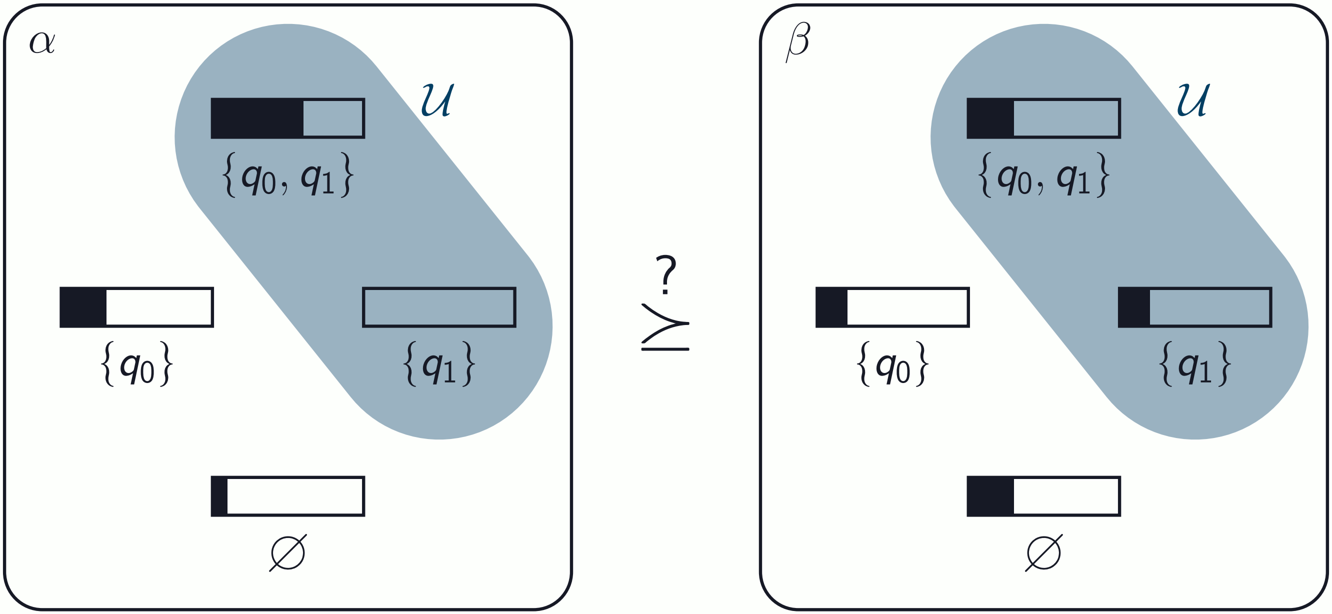

$\sum_{R\in\mathcal{U}} \alpha(R) = 0.6 + 0.0 = 0.6$

$0.5 = 0.3 + 0.2 = \sum_{R\in\mathcal{U}} \beta(R)$

Order on $\mathcal{D}(\mathsf{P}(Q))$

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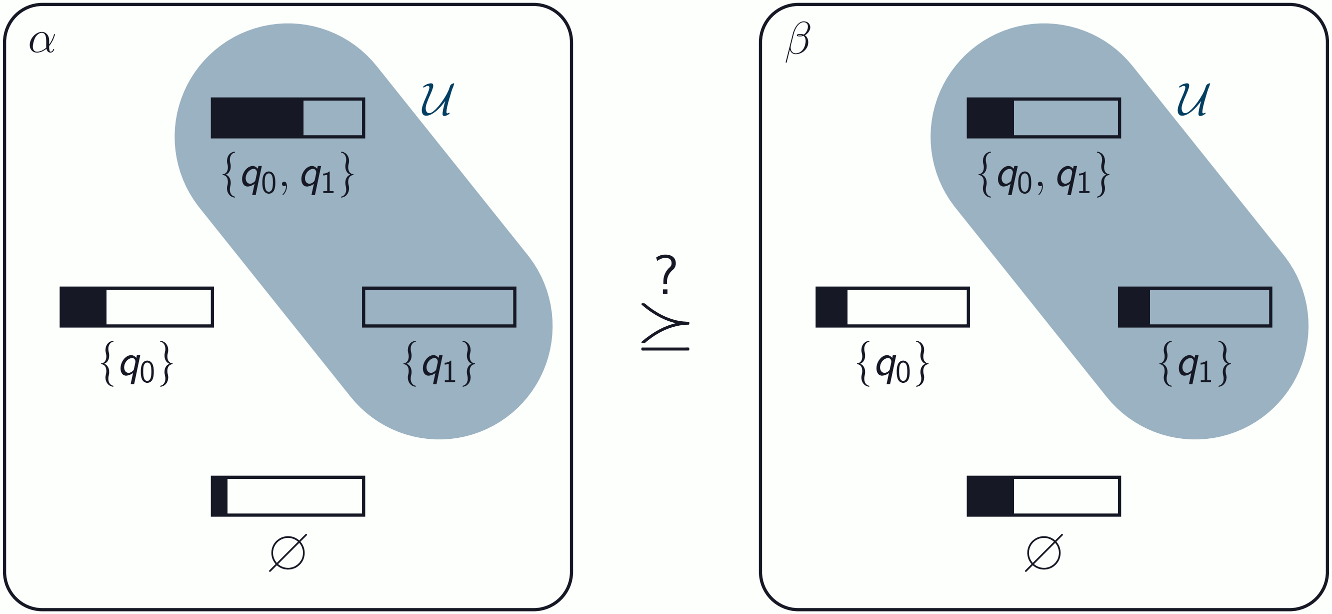

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Order on $\mathcal{D}(\mathsf{P}(Q))$

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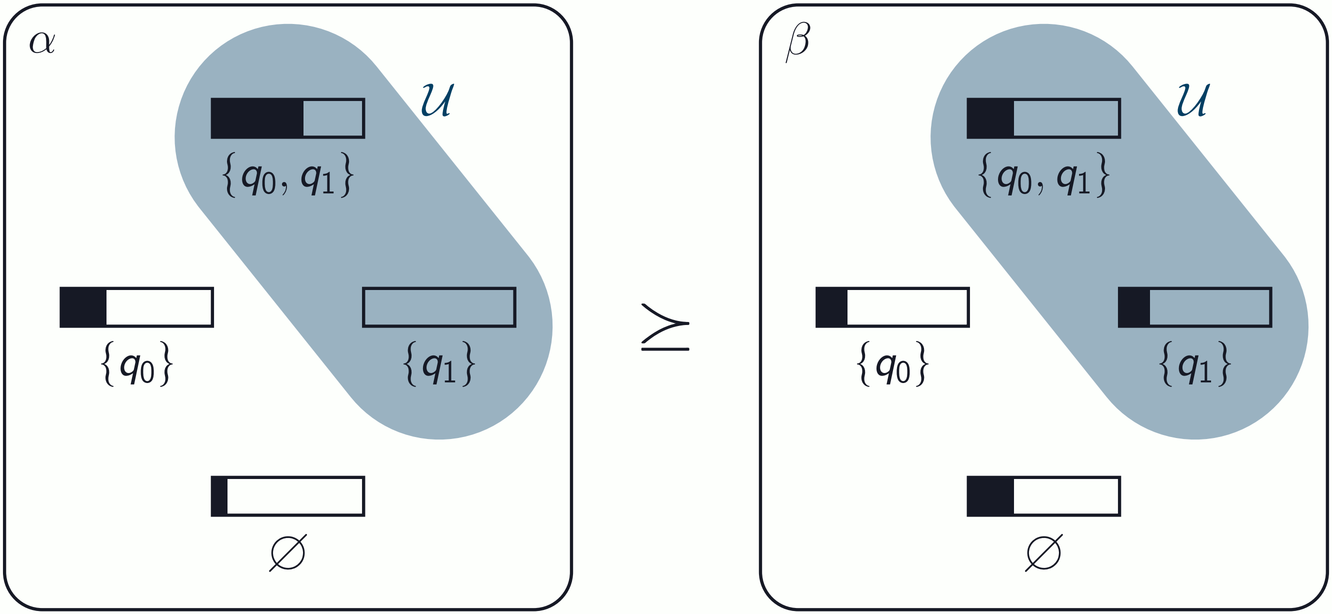

$$\sum_{R\in\mathcal{U}} \alpha(R) = 0.6 + 0.0 = 0.6 \quad \geqslant \quad 0.5 = 0.3 + 0.2 = \sum_{R\in\mathcal{U}} \beta(R)$$

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"**probabilistic powerdomains**"

(Saheb-Djahromi [1980])

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using **First-order** theory of reals (Tarski [1951])

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