Computing measures of weak-MSO definable sets of trees

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Abstract

This work addresses the problem of computing measures of recognisable sets of infinite trees. An algorithm is provided to compute the probability measure of a tree language recognisable by a weak alternating automaton, or equivalently definable in weak monadic second-order logic. The measure is the uniform coin-flipping measure or more generally it is generated by a branching stochastic process. The class of tree languages in consideration, although smaller than all regular tree languages, comprises in particular the languages definable in the alternation-free μ-calculus or in temporal logic CTL. Thus, the new algorithm may enhance the toolbox of probabilistic model checking.

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1 Introduction

The non-emptiness problem asks if an automaton accepts at least one object. From a logical perspective, it is an instance of the consistency question: does a given specification have a model? Sometimes it is also relevant to ask a quantitative version of this question: whether a *non-negligible* set of models satisfy the specification. When taken to the realm of probability theory, this boils down to estimating the probability that a random object is accepted by a given automaton. In this paper, models under consideration are infinite binary trees labelled by a finite alphabet. Our main problem of interest is the following.

▷ Problem 1. Given a regular tree language L, compute the probability that a randomly generated tree belongs to L.

In other words, we ask for the probability measure of L. Here, the tree language L might be given by a formula of monadic second-order logic, but for complexity reasons it is more suitable to present it by a tree automaton or by a formula of modal μ-calculus, see e.g. [9, 13]. By default, the considered measure is the uniform *coin-flipping* measure, where each letter is chosen independently at random; but also more specific measures are of interest. If the computed probability is rational then it can be represented explicitly, but the measure can be irrational, see e.g. [15], and may require more complex representation. One of the possible choices, exploited in this paper, is a formula over the field of reals $\mathbb{R}$.

Chen et al. [6] addressed Problem 1 in the case where the tree language L is recognised by a deterministic top-down automaton and the measure is induced by a stochastic branching process, which then makes also a part of the input data. Their algorithm compares the probability with any given rational number in polynomial space and with 0 or 1 in polynomial time. The limitations of this result come from the deterministic nature of the considered automata: deterministic top-down tree automata are known to have limited expressive power within all regular tree languages.

Michalewski and Mio [15] stated Problem 1 explicitly and solved it for languages L given by so-called *game automata* and the coin-flipping measure. This class of automata subsumes deterministic ones and captures some important examples including the game languages, cf. [10], but even here the strength of non-determinism is limited; in particular, the class is not closed under finite union. The algorithm from [15] reduces the problem to computing the value of a Markov branching play, and uses Tarski's decision procedure for the theory of reals. These authors also discover that the measure of a regular tree language can be irrational, which stays in contrast with the case of ω-regular languages, i.e. regular languages of infinite words, where the coin-flipping measure is always rational, cf. [5].

Another step towards a solution to Problem 1 was made by the second author of the present article, who proposed an algorithm to compute the coin-flipping measure of tree languages definable in fragments of first-order logic [20]. This work is subsumed in a report [21] (accepted for publication in a journal) co-authored with the third author, where a new class of languages L is also resolved: tree languages recognised by safety automata, i.e. non-deterministic automata with a trivial accepting condition.

An analogue of Problem 1 can be stated for ω-regular languages. As noted by [6], the problem then reduces to a well-known question in verification solved by Courcoubetis and Yannakakis [8] already in the 1990s, namely whether a run of a finite-state Markov chain satisfies an ω-regular property. The algorithm runs in single-exponential time w.r.t. the automaton (and linear w.r.t. the Markov chain). A related question was also studied by Staiger [24], who gave an algorithm to compute Hausdorff dimension and Hausdorff measure of a given ω-regular language.

In general, Problem 1 remains unsolved. At first sight, one may even wonder if it is well-stated, as regular tree languages need not in general be Borel, cf. [18]. However, due to [12, 16], we know that regular languages of trees are always universally measurable.

In the present paper, we solve Problem 1 in the case where the language L is recognised by a weak alternating automaton or, equivalently, defined by a formula of weak monadic second-order logic, cf. [17]. The class of tree languages in consideration is incomparable with the one considered by Michalewski and Mio [15], but subsumes those considered in [20, 21]. Yet another presentation of this class can be given in terms of alternation-free fragment of modal μ-calculus, see [1] for details. This fragment is known to be useful in verification and model checking, in particular, temporal logic CTL embeds into this fragment.

We consider the coin-flipping measure as our primary case, but we also show how to extend our approach to measures generated by stochastic branching processes, as in [6]. The computed probability is presented by a first-order formula in prenex normal form over the field of reals. The provided formula is exponential in the size of the automaton and polynomial in the size of the branching process. Moreover, the quantifier alternation of the computed formula is constant (equal 4). Combined with the known decision procedures for the theory of reals, this gives the following.

▶ **Theorem 2.** *There is an algorithm that inputs a weak alternating parity automaton $\mathcal{A}$, a branching process $\mathcal{P}$, and a rational number q encoded in binary; and decides if the measure generated by $\mathcal{P}$ of the language recognised by $\mathcal{A}$ is equal, smaller, or greater than q. The algorithm works in time polynomial in q, doubly exponential in $\mathcal{A}$, and singly exponential in $\mathcal{P}$.*

Similarly to the approach taken in [21], we reduce the problem to computation of an appropriate probability distribution over the powerset of the automaton's states. To do so, we consider the set of all such distributions $\mathcal{D}\mathsf{P}(Q)$ with a suitable ordering $\preceq$. The structure is in fact a finitary case of a probabilistic powerdomain introduced by Saheb-Djahromi [22] (see also [14]), but we do not exploit category-theoretic concepts in this paper. The key step is an approximation of the target language L by two families of tree languages representing safety and reachability properties, respectively. Then we can apply fixed-point constructions thanks to a kind of synergy between the order and topological properties of $\mathcal{D}\mathsf{P}(Q)$.

2 Trees, topology, and measure

The set of natural numbers $\{0, 1, 2, \dots\}$ is denoted by $\mathbb{N}$, or by ω whenever we treat it as an ordinal. A finite non-empty set A is called an *alphabet*. By $\mathsf{P}(X)$ we denote the family of all subsets of a set X. The set of *finite words* over an alphabet A (including the *empty word* ε) is denoted by A^*, and the set of ω-words by A^ω. The *length* of a finite word $w \in A^*$ is denoted by $|w|$. A *full infinite binary tree* over an alphabet A (or simply a *tree* if confusion does not arise) is a mapping $t\colon \{\mathtt{L}, \mathtt{R}\}^* \to A$. The set of all such trees, denoted by Tr_A, can be equipped with a topology induced by a metric

$$d(t_1, t_2) = \begin{cases} 0 & \text{if } t_1 = t_2 \\ 2^{-n} \text{ with } n = \min\{|w| \mid t_1(w) \neq t_2(w)\} & \text{otherwise.} \end{cases}$$

It is well-known that this topology coincides with the product topology on A^ω, where A is a discrete topological space. The topology can be generated by a basis consisting of all the sets U_f, where $f\colon \mathrm{dom}(f) \to A$ is a function with a finite domain $\mathrm{dom}(f) \subset \{\mathtt{L}, \mathtt{R}\}^*$, and U_f consists of all trees t that coincide with f on $\mathrm{dom}(f)$. If A has at least 2 elements then this topology is homeomorphic to the Cantor discontinuum $\{0, 1\}^\omega$ (see, e.g. [19]).

The set of trees can be further equipped with a probabilistic measure μ_0, which is the standard Lebesgue measure on the product space defined on the basis by $\mu_0(U_f) = |A|^{-|\mathrm{dom}(f)|}$.

We note a useful property of this measure, which intuitively amounts to saying that events happening in incomparable nodes are independent. For $t \in \mathrm{Tr}_A$ and $v \in \{\mathtt{L},\mathtt{R}\}^*$, the subtree of t induced by v is a tree $t{\restriction_v} \in \mathrm{Tr}_A$ defined by $t{\restriction_v}(w) = t(vw)$, for $w \in \{\mathtt{L},\mathtt{R}\}^*$.

▶ Remark 3. If $v_1, \ldots, v_k \in \{\mathtt{L},\mathtt{R}\}^*$ are pairwise incomparable nodes (i.e., none is a prefix of another) and $V_1, \ldots, V_k \subseteq \mathrm{Tr}_A$ are Borel sets then

$$\mu_0\left(\{t \in \mathrm{Tr}_A \mid t{\restriction_{v_i}} \in V_i \text{ for } i = 1, \ldots, k\}\right) \quad = \quad \mu_0(V_1) \cdot \ldots \cdot \mu_0(V_k). \tag{1}$$

We refer to e.g. [12] for more detailed considerations of measures on sets of infinite trees.

3 Tree automata and games

An *alternating parity automaton* over infinite trees can be presented as a tuple $\mathcal{A} = \langle A, Q, q_{\mathrm{I}}, \delta, \Omega\rangle$, where A is a finite alphabet; Q a finite set of *states*; $q_{\mathrm{I}} \in Q$ an *initial state*; $\delta\colon Q \times A \to \mathrm{BC}^+\left(\{\mathtt{L},\mathtt{R}\} \times Q\right)$ a *transition function* that assigns to a pair $(q,a) \in Q \times A$ a finite positive Boolean combination of pairs $(d,q') \in \{\mathtt{L},\mathtt{R}\} \times Q$; and finally $\Omega\colon Q \to \mathbb{N}$ is a *priority mapping*.

In this paper, we assume that automata are *weak*, i.e. the priorities $\Omega(q)$ are non-increasing along transitions. More precisely, if (d,q') is an atom that appears in the formula $\delta(q,a)$ then $\Omega(q) \geq \Omega(q')$. Given $n \in \mathbb{N}$, we denote by $Q_{<n}$ and $Q_{\geq n}$ the subsets of Q consisting of those states whose priority is respectively strictly smaller or greater than n.

The semantics of an automaton can be given in a terms of a game played by two players $\exists$ and $\forall$ over a tree t in Tr_A from a state $p \in Q$. Let Γ be the set of all sub-formulae of the formulae in $\delta(q,a)$, for all $(q,a) \in Q \times A$. The set of *positions* of the game is the set $\left(Q \sqcup \Gamma\right) \times \{\mathtt{L},\mathtt{R}\}^*$ and the *initial position* is $\left(p, \varepsilon\right)$. The positions of the form $\left(q,v\right)$, $\left(\phi_1{\vee}\phi_2, v\right)$, and $\left((d,q),v\right)$ are controlled by $\exists$, while the positions of the form $\left(\phi_1{\wedge}\phi_2, v\right)$ are controlled by $\forall$. The edges connect the following types of positions:

- $\left(q,v\right)$ and $\left(\delta(q,t(v)),v\right)$,
- $\left(\phi_1{\vee}\phi_2,v\right)$ and $\left(\phi_i,v\right)$ for $i = 1,2$,
- $\left(\phi_1{\wedge}\phi_2,v\right)$ and $\left(\phi_i,v\right)$ for $i = 1,2$,
- $\left((d,q),v\right)$ and $\left(q, v \cdot d\right)$.

We assume that every formula in the image $\delta(Q \times A)$ is non-trivial and, thus, every position is a source of some edge.

The directed graph described above forms the *arena* of our game that we denote by $\mathcal{G}(t,p)$. A *play* in the arena is any infinite path starting from the initial position $\left(p,\varepsilon\right)$. We call the positions of the form (q,v) *state positions*. Given a play π, the *states* of the play denoted $\mathrm{states}(\pi)$ is the sequence of states $(q_0, q_1, \ldots) \in Q^\omega$ such that the successive state positions visited during π are $\left(q_i, v_i\right)$, for $i = 0, 1, \ldots$, and some $(v_i)_{i\in\omega}$.

To complete the definition of the game, we specify a winning criterion for $\exists$. The default is the *parity condition*, but we will also consider other criteria. Let

$$\mathrm{Runs} \stackrel{\text{def}}{=} \left\{\rho \in Q^\omega \mid \forall i \in \omega.\ \Omega(\rho(i)) \geq \Omega(\rho(i{+}1))\right\}$$

be the set that contains all sequences of states that induce non-increasing sequences of priorities. Notice that since $\mathcal{A}$ is weak, only such sequences may arise in the game. In general, a *winning condition* is any set $W \subseteq \mathrm{Runs}$. That is, a play π is *winning* for $\exists$ with respect to W if, and only if, states $(\pi) \in W$. The game with a winning set W is denoted by $\mathcal{G}(t, p, W)$.

The *parity condition* $W_P \subseteq \mathrm{Runs}$ for a weak automaton amounts to: $(q_0, q_1, \ldots) \in W_P$ if $\lim_{i\to\infty} \Omega(q_i) \equiv 0 \mod 2$, i.e. the limit priority of states visited in a play is even. Let $\mathrm{L}(\mathcal{A}, p)$ be the set of trees such that $\exists$ has a winning strategy in $\mathcal{G}(t, p, W_P)$. Then, the language of an automaton $\mathcal{A}$ is the set $\mathrm{L}(\mathcal{A}) \stackrel{\mathrm{def}}{=} \mathrm{L}(\mathcal{A}, q_{\mathrm{I}})$, where q_{I} is the initial state of $\mathcal{A}$.

As mentioned above, we will consider games with various winning criteria. The following simple observation is useful.

▶ Remark 4. If $W \subseteq W' \subseteq \mathrm{Runs}$ then the following implication holds: if $\exists$ wins $\mathcal{G}(t, p, W)$ then $\exists$ wins also $\mathcal{G}(t, p, W')$.

Since the winning criteria in consideration will always be ω-regular languages of infinite words, we implicitly rely on the following classical fact (cf. [13]).

▶ **Proposition 5.** *Games on graphs with ω-regular winning conditions are finite memory determined.*

We will also use the following fact, cf. e.g. [17, 23].

▶ **Proposition 6.** *For a weak alternating parity automaton $\mathcal{A}$, all tree languages $\mathrm{L}(\mathcal{A}, p)$ are Borel and, consequently, measurable with respect to the uniform measure μ_0 (and also any other Borel measure on Tr_A).*

Note that measurability holds for non-weak automata as well [12].

4 Approximations

For the sake of this section we fix a weak alternating parity automaton $\mathcal{A}$. Our aim is to provide some useful approximations for the tree languages $\mathrm{L}(\mathcal{A}, p)$. The approximations are simply some families of tree languages indexed by states $p \in Q$. Those families, called *Q-indexed families*, or *Q-families* for short, are represented by functions $\mathcal{L}\colon Q \to \mathsf{P}(\mathrm{Tr}_A)$. By the construction, we will guarantee that the tree languages $\mathcal{L}(q)$ will themselves be recognisable by some weak alternating automata. Each Q-family naturally possesses a dual representation by a mapping $\mathrm{Tr}_A \to \mathsf{P}(Q)$ that we denote by the same letter (but with different brackets)

$$\mathcal{L}[t] \stackrel{\mathrm{def}}{=} \{q \in Q \mid t \in \mathcal{L}(q)\} \in \mathsf{P}(Q).$$

If $\rho \in \mathrm{Runs} \subseteq Q^\omega$ is an infinite sequence of states then $\lim_{i\to\infty} \Omega(\rho(i))$ (denoted by $\mathrm{limit}(\rho)$) exists, because by the definition of Runs the priorities are non-increasing and bounded. Recall that $W_P \subseteq \mathrm{Runs}$ is the set of runs satisfying the parity condition, i.e. $W_P = \{\rho \in \mathrm{Runs} \mid \mathrm{limit}(\rho) \equiv 0 \mod 2\}$. For $i, n \in \mathbb{N}$, consider the following subsets of Runs:

$$S_i^n \stackrel{\mathrm{def}}{=} W_P \cup \big\{\rho \in \mathrm{Runs} \mid \Omega(\rho(i)) \geq n\big\},$$
$$S_\infty^n \stackrel{\mathrm{def}}{=} W_P \cup \big\{\rho \in \mathrm{Runs} \mid \mathrm{limit}(\rho) \geq n\big\},$$
$$R_i^n \stackrel{\mathrm{def}}{=} W_P \cap \big\{\rho \in \mathrm{Runs} \mid \Omega(\rho(i)) < n\big\},$$
$$R_\infty^n \stackrel{\mathrm{def}}{=} W_P \cap \big\{\rho \in \mathrm{Runs} \mid \mathrm{limit}(\rho) < n\big\}.$$

Connotatively, the name of the sets S_i^n comes from the condition of *safety*, while the sets R_i^n are named after *reachability*. More precisely, S_i^n is an over-approximation of W_P, that makes ∃ win also if she manages to reach a priority $\geq n$ in the ith visited node of a given tree. Analogously, R^n_i is an under-approximation of W_P that makes ∀ win in the above case.

Based on the above definitions, we define the respective Q-families. For $p \in Q$, let $\mathcal{S}^n_i(p)$, $\mathcal{S}^n_\infty(p)$, $\mathcal{R}^n_i(p)$, and $\mathcal{R}^n_\infty(p)$ be the sets of trees such that ∃ has a winning strategy in the game $\mathcal{G}(t, p, W)$, where W is respectively S^n_i, S^n_∞, R^n_i, and R^n_∞. Figure 1 below depicts the way these Q-families are used in the general construction.

It is easy to see that all the tree languages above can be recognised by weak parity alternating automata.

▶ **Lemma 7.** *For every $n \in \mathbb{N}$ and $i \in \mathbb{N}$, we have*

$$S^n_i \supseteq S^n_{i+1} \supseteq S^n_\infty \text{ and } R^n_i \subseteq R^n_{i+1} \subseteq R^n_\infty.$$

Analogously, for every $p \in Q$,

$$\mathcal{S}^n_i(p) \supseteq \mathcal{S}^n_{i+1}(p) \supseteq \mathcal{S}^n_\infty(p) \text{ and } \mathcal{R}^n_i(p) \subseteq \mathcal{R}^n_{i+1}(p) \subseteq \mathcal{R}^n_\infty(p).$$

Proof. The first property follows directly from the definition of Runs, which guarantees that $\Omega(\rho(0)) \geq \Omega(\rho(1)) \geq \ldots \geq \text{limit}(\rho)$. Then, the second property follows from Remark 4. ◀

It is straightforward to see that $S^n_\infty = \bigcap_{i\in\mathbb{N}} S^n_i$ and $R^n_\infty = \bigcup_{i\in\mathbb{N}} R^n_i$. However, it is not clear that these equalities imply the desired properties for the respective sets of trees. Lemma 9 below implies that it is the case. The proof relies on combinatorics of binary trees, namely on König's Lemma.

▶ **Lemma 8.** *Take $n \in \mathbb{N}$ and $p \in Q$. Let $B^n_\infty = \{\rho \in \text{Runs} \mid \text{limit}(\rho) < n\}$ and, for $i \in \mathbb{N}$, let $B^n_i = \{\rho \in \text{Runs} \mid \Omega(\rho(i)) < n\}$. If σ is a winning strategy of ∃ in $\mathcal{G}(t, p, B^n_\infty)$ then there exists a number $J \in \mathbb{N}$, such that σ is actually winning in $\mathcal{G}(t, p, B^n_J)$. An analogous property holds if σ is a winning strategy for ∀.*

Proof. Let σ be a winning strategy of ∃ in $\mathcal{G}(t, p, B^n_\infty)$ (the case of ∀ is completely analogous). Let $T \subseteq \big(Q \times \{\mathtt{L}, \mathtt{R}\}\big)^*$ be the set of sequences $(q_i, d_i)_{i\leq\ell}$, with $q_0 = p$, such that there exists a play consistent with σ that visits all the positions $(q_i, d_0 \cdots d_{i-1})$ for $i = 0, 1, \ldots, \ell$, and additionally $\Omega(q_\ell) \geq n$. Observe that T is prefix-closed. Thus, we can treat T as a tree. Moreover, as $Q \times \{\mathtt{L}, \mathtt{R}\}$ is finite, T is finitely branching. If T is finite then there exists J such that all the sequences in T have length at most J. In that case σ is winning in $\mathcal{G}(t, p, B^n_J)$, and we are done.

For the sake of contradiction, suppose that T is infinite. Apply König's Lemma to obtain an infinite path $(q_i, d_i)_{i\in\omega}$ in T. By the definition of T, it implies that there exists an infinite play consistent with σ such that $(q_i)_{i\in\omega}$ is the sequence of states visited during the play. But this is a contradiction, because $\text{limit}\big((q_i)_{i\in\omega}\big) \geq n$ by the definition of T and, therefore, the considered play is losing for ∃. ◀

▶ **Lemma 9.** *Using the above notions, for every state $p \in Q$, we have*

$$\mathcal{S}^n_\infty(p) = \bigcap_{i\in\mathbb{N}} \mathcal{S}^n_i(p) \quad \text{and} \quad \mathcal{R}^n_\infty(p) = \bigcup_{i\in\mathbb{N}} \mathcal{R}^n_i(p).$$

Proof. Consider the first claim and take a tree $t \in \text{Tr}_A$ such that for every $i \in \mathbb{N}$ we have $t \in \mathcal{S}^n_i(p)$. We need to prove that $t \in \mathcal{S}^n_\infty(p)$. Assume contrarily, that $t \notin \mathcal{S}^n_\infty(p)$. By determinacy, see Proposition 5, it means that there exists a strategy σ' of ∀ such that for every play π consistent with σ', we have $\text{limit}(\text{states}(\pi)) < n$ and $\text{limit}(\text{states}(\pi))$ is odd. Hence, in particular, σ' is winning for ∀ in $\mathcal{G}(t, p, B^n_\infty)$. Therefore, by Lemma 8, we know that there exists a number $J \in \mathbb{N}$ such that, for every π consistent with σ' with states$(\pi) = (q_0, q_1, \ldots)$, we have $\Omega(q_J) < n$. Therefore, the strategy σ' witnesses that $t \notin \mathcal{S}^n_J(p)$, a contradiction.

We now prove the second claim. Take a tree $t \in \mathcal{R}^n_\infty(p)$. We need to prove that $t \in \mathcal{R}^n_i(p)$ for some $i \in \mathbb{N}$. Let σ be a strategy of $\exists$ witnessing that $t \in \mathcal{R}^n_\infty(p)$. Again, Lemma 8 guarantees that there exists a number $J \in \mathbb{N}$ such that if π is a play consistent with σ with states$(\pi) = (q_0, q_1, \ldots)$ then $\Omega(q_J) < n$. Thus, $t \in \mathcal{R}^n_J(p)$. ◀

The following lemma provides another characterisation of the above Q-families.

▶ **Lemma 10.** *For each $p \in Q$, we have $\mathcal{S}^0_i(p) = \mathrm{Tr}_A$ and $\mathcal{R}^0_i(p) = \emptyset$. Take $n > 0$. If $\Omega(p) \geq n$ then $\mathcal{S}^n_0(p) = \mathrm{Tr}_A$ and $\mathcal{R}^n_0(p) = \emptyset$. If $\Omega(p) < n$ then*

$$\mathrm{L}(\mathcal{A}, p) = \mathcal{S}^n_0(p), \qquad \mathrm{L}(\mathcal{A}, p) = \mathcal{R}^n_0(p),$$
$$\mathrm{L}(\mathcal{A}, p) = \mathcal{S}^{n-1}_\infty(p) \quad \textit{for odd } n, \qquad \mathrm{L}(\mathcal{A}, p) = \mathcal{R}^{n-1}_\infty(p) \quad \textit{for even } n.$$

Proof. The cases of $n = 0$ are trivial. The first two claims in the case $\Omega(p) \geq n$ follow directly from the definitions. Take p such that $\Omega(p) < n$. Notice that in that case the sequence of states ρ in a play in $\mathcal{G}(t, p)$ satisfies

$$\rho \in W_P \iff \rho \in S^n_0, \qquad \rho \in W_P \iff \rho \in R^n_0,$$
$$\rho \in W_P \iff \rho \in S^{n-1}_\infty \quad \text{for odd } n, \qquad \rho \in W_P \iff \rho \in R^{n-1}_\infty \quad \text{for even } n.$$

where the first two equivalences follow from the fact that $\Omega(\rho(0)) = \Omega(p) < n$. The last two equivalences can be derived from the fact that $\mathrm{limit}(\rho) \leq \Omega(p) < n$. First, we have $\mathrm{limit}(\rho) \geq n{-}1 \Leftrightarrow \mathrm{limit}(\rho) = n{-}1$. Thus, if n is odd and $\mathrm{limit}(\rho) \geq n{-}1$ then we know that $\mathrm{limit}(\rho)$ is even. Analogously, if n is even then $n{-}1$ is odd and the fact that $\mathrm{limit}(\rho)$ is even guarantees that $\mathrm{limit}(\rho) < n{-}1$.

Clearly, the above equivalences imply that, under the assumption of the lemma, a strategy winning for the condition W_P is winning for the respective conditions and vice-versa. ◀

Our aim now is to define a function $\Delta\colon \mathsf{P}(Q) \times A \times \mathsf{P}(Q) \to \mathsf{P}(Q)$ that will allow us to form equations over Q-families. An ordered pair of sets of states $P_{\mathtt{L}}, P_{\mathtt{R}} \in \mathsf{P}(Q)$ induces a *valuation* $v_{P_{\mathtt{L}}, P_{\mathtt{R}}}$ to the atoms in $\{\mathtt{L}, \mathtt{R}\} \times Q$ defined by: $v_{P_{\mathtt{L}}, P_{\mathtt{R}}}(d, p)$ is true if $p \in P_d$. Now, consider additionally a letter $a \in A$ and put

$$\Delta(P_{\mathtt{L}}, a, P_{\mathtt{R}}) = \big\{q \in Q \mid v_{P_{\mathtt{L}}, P_{\mathtt{R}}} \models \delta(q, a)\big\}.$$

Equivalently, $q \in \Delta(P_{\mathtt{L}}, a, P_{\mathtt{R}})$ if $\exists$ can play the finite game represented by $\delta(q, a)$ in such a way to reach only such atoms (d, p) that satisfy $p \in P_d$.

▶ **Lemma 11.** *The function $\Delta\colon \mathsf{P}(Q) \times A \times \mathsf{P}(Q) \to \mathsf{P}(Q)$ is monotone, i.e. if $P_{\mathtt{L}} \subseteq P'_{\mathtt{L}}$ and $P_{\mathtt{R}} \subseteq P'_{\mathtt{R}}$ then $\Delta(P_{\mathtt{L}}, a, P_{\mathtt{R}}) \subseteq \Delta(P'_{\mathtt{L}}, a, P'_{\mathtt{R}})$.*

Proof. It follows directly from the fact that the Boolean formulae in $\delta(q, a)$ are positive. ◀

Recall that $t{\restriction}_v \in \mathrm{Tr}_A$ denotes the subtree of t induced by a node v, cf. Section 2. The following lemma shows how to increase the index i of the above Q-families $\mathcal{S}^n_i$ and $\mathcal{R}^n_i$.

▶ **Lemma 12.** *Take $n \in \mathbb{N}$, $i \in \mathbb{N}$, and a tree $t \in \mathrm{Tr}_A$. Then we have:*

$$\mathcal{S}^n_{i+1}[t] = \Delta\big(\mathcal{S}^n_i[t{\restriction}_{\mathtt{L}}], t(\varepsilon), \mathcal{S}^n_i[t{\restriction}_{\mathtt{R}}]\big),$$
$$\mathcal{R}^n_{i+1}[t] = \Delta\big(\mathcal{R}^n_i[t{\restriction}_{\mathtt{L}}], t(\varepsilon), \mathcal{R}^n_i[t{\restriction}_{\mathtt{R}}]\big).$$

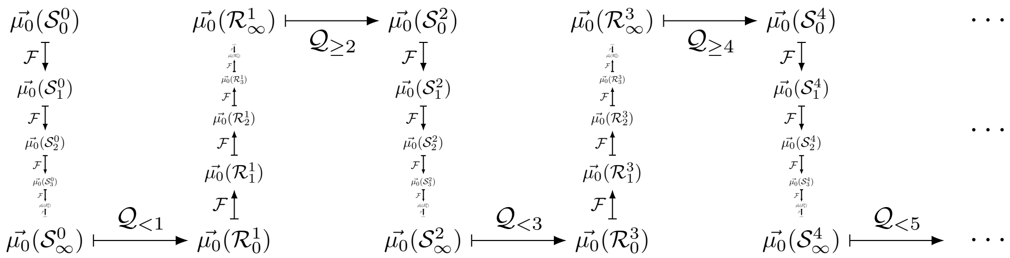

Figure 1 A schematic presentation of the relationship between the distributions used in the proof. The vertical axis represents the order $\preceq$, i.e. $\vec{\mu_0}(\mathcal{S}_0^0) \succeq \vec{\mu_0}(\mathcal{S}_\infty^0)$. The edges marked $\mathcal{F}$, $\mathcal{Q}_{<n}$, and $\mathcal{Q}_{\geq n}$ represent applications of the respective operations. The vertical convergence is understood in terms of pointwise limits in $\mathbb{R}^{\mathsf{P}(Q)}$.

The proof of this lemma is based on a standard technique of merging strategies: the game $\mathcal{G}(t,p)$ can be split into a finite game corresponding to the formula $\delta\big(p, t(\varepsilon)\big)$ that leads to the sub-games $\mathcal{G}(t{\restriction}_{\mathtt{L}}, p_{\mathtt{L}})$ and $\mathcal{G}(t{\restriction}_{\mathtt{R}}, p_{\mathtt{R}})$ for some states $p_{\mathtt{L}}, p_{\mathtt{R}} \in Q$.

Proof. Take a play π in the arena $\mathcal{G}(t,p)$ for some state $p \in Q$. Recall that, by the definition of the game, the initial position of the play is (p,ε) and the next state position will have the form (q,d), for some $q \in Q$ and $d \in \{\mathtt{L},\mathtt{R}\}$. Consider the suffix of the play π starting from that position. Clearly, this suffix induces a play, say π', in the arena $\mathcal{G}(t{\restriction}_d, q)$, starting from the position (q,ε) (technically, to satisfy our definition, we need also to replace every position (α, dv) by (α, v) in the original play). Moreover, the sequence of states visited by π', states(π'), is a suffix of the sequence states(π) obtained by removing just the first element. By the definition of S_i^n and R_i^n we have therefore

$$\text{states}(\pi) \in S_{i+1}^n \iff \text{states}(\pi') \in S_i^n, \text{ and states}(\pi) \in R_{i+1}^n \iff \text{states}(\pi') \in R_i^n. \quad (2)$$

We will now provide the proof for $\mathcal{S}_{i+1}^n$, the case of $\mathcal{R}_{i+1}^n$ is analogous. Let $P_{\mathtt{L}}$ and $P_{\mathtt{R}}$ equal respectively $\mathcal{S}_i^n[t{\restriction}_{\mathtt{L}}]$ and $\mathcal{S}_i^n[t{\restriction}_{\mathtt{R}}]$. Put $a = t(\varepsilon)$. Recall that by the duality of the two representations of Q-families, $p \in \mathcal{S}_{i+1}^n[t]$ iff $t \in \mathcal{S}_{i+1}^n(p)$. So we need to prove that for every $p \in Q$ we have $t \in \mathcal{S}_{i+1}^n(p)$ if and only if $p \in \Delta(P_{\mathtt{L}}, a, P_{\mathtt{R}})$.

Assume that $t \in \mathcal{S}_{i+1}^n(p)$. Take a strategy σ witnessing that. Notice that if a position of the form (q,d) can be reached by σ then by (2) we know that $t{\restriction}_d \in \mathcal{S}_i^n(q)$, i.e. $q \in P_d$. Thus, the strategy σ witnesses that $p \in \Delta(P_{\mathtt{L}}, a, P_{\mathtt{R}})$.

For the opposite direction, assume that $p \in \Delta(P_{\mathtt{L}}, a, P_{\mathtt{R}})$. This means that there exists a finite strategy of $\exists$ that allows her to resolve the formula $\delta(p,a)$ in such a way that for every atom (d,q) that can be reached by this strategy, we have $(d,q) \in P_d$. The last means that $\exists$ has a winning strategy in the game $\mathcal{G}(t{\restriction}_d, q, \mathcal{S}_i^n)$. Now we can combine all above strategies in a strategy in the game $\mathcal{G}(t,p,S_{i+1}^n)$, which by Equation (2) is again winning for $\exists$. Hence, $t \in \mathcal{S}_{i+1}^n(p)$, as desired. ◀

5 Measures and distributions

Following an approach started in [21], we transfer the problem of computing measures of tree languages $\mathrm{L}(\mathcal{A},p)$ to computing a suitable probability distribution on the sets of the automaton states. We start with a general construction. For a finite set X, consider the set of probability distributions over X, $\mathcal{D}X \stackrel{\text{def}}{=} \big\{\alpha\colon X \to [0,1] \mid \sum_{x\in X} \alpha(x) = 1\big\}$. Observe that, if X is partially ordered by a relation $\leq$ then $\mathcal{D}X$ is partially ordered by a relation $\preceq$ defined as follows: $\alpha \preceq \beta$ if for every upward-closed[1] set $U \subseteq X$, we have $\sum_{x\in U} \alpha(x) \leq \sum_{x\in U} \beta(x)$. In this article, we are interested in $\langle X, \leq\rangle$ being the powerset $\mathsf{P}(Q)$ ordered by inclusion $\subseteq$.

▶ Remark 13. The relation $\preceq$ is a partial order on $\mathcal{D}X$ (as an intersection of a finite family of partial orders).

Every Q-family $\mathcal{L}$ for a weak alternating automaton $\mathcal{A}$ induces naturally a member of $\mathcal{D}\mathsf{P}(Q)$, which is a distribution $\vec{\mu_0}\big(\mathcal{L}\big)$ defined by

$$\vec{\mu_0}\big(\mathcal{L}\big)(P) = \mu_0\big\{t \in \mathrm{Tr}_A \mid \mathcal{L}[t] = P\big\}.$$

Here μ_0 is the uniform probability measure on Tr_A. The sets in consideration are measurable thanks to Proposition 6.

Note that if the language family is exactly $\mathcal{L}(q) = \mathrm{L}(\mathcal{A}, q)$ then the probability assigned to a set of states P amounts to the probability that a randomly chosen tree, with respect to μ_0, is accepted precisely from the states in P.

▶ **Lemma 14.** *If for each $q \in Q$ we have $\mathcal{L}(q) \subseteq \mathcal{L}'(q)$ then $\vec{\mu_0}(\mathcal{L}) \preceq \vec{\mu_0}(\mathcal{L}')$ in $\mathcal{D}\mathsf{P}(Q)$.*

Proof. Take any upward-closed family $U \subseteq \mathsf{P}(Q)$. Then

$$\sum_{P\in U} \vec{\mu_0}\big(\mathcal{L}\big)(P) = \sum_{P\in U} \mu_0\big\{t \in \mathrm{Tr}_A \mid \mathcal{L}[t] = P\big\} = \mu_0\big\{t \in \mathrm{Tr}_A \mid \mathcal{L}[t] \in U\big\} \leq$$

$$\leq \mu_0\big\{t \in \mathrm{Tr}_A \mid \mathcal{L}'[t] \in U\big\} = \sum_{P\in U} \mu_0\big\{t \in \mathrm{Tr}_A \mid \mathcal{L}'[t] = P\big\} = \sum_{P\in U} \vec{\mu_0}\big(\mathcal{L}'\big)(P),$$

where the middle inequality follows from the assumption that $\mathcal{L}(q) \subseteq \mathcal{L}'(q)$ and the fact that the family U is upward-closed. ◀

We now examine the sequences of distributions $\vec{\mu_0}\big(\mathcal{S}^n_i\big)$, $\vec{\mu_0}\big(\mathcal{R}^n_i\big)$, $\vec{\mu_0}\big(\mathcal{S}^n_\infty\big)$, and $\vec{\mu_0}\big(\mathcal{R}^n_\infty\big)$ arising from the Q-families introduced in the previous section. Our aim is to bind them by equations computable within $\mathcal{D}\mathsf{P}(Q)$. As an analogue to the operation Δ, we introduce a function $\mathcal{F}\colon \mathcal{D}\mathsf{P}(Q) \to \mathcal{D}\mathsf{P}(Q)$ defined for $\beta \in \mathcal{D}\mathsf{P}(Q)$ and $P \in \mathsf{P}(Q)$ by

$$\mathcal{F}(\beta)(P) = \frac{1}{|A|} \cdot \sum_{(P_{\mathtt{L}}, a, P_{\mathtt{R}})\in\Delta^{-1}(P)} \beta(P_{\mathtt{L}}) \cdot \beta(P_{\mathtt{R}}). \tag{3}$$

Note that the formula guarantees that $\mathcal{F}(\beta)$ is indeed a probabilistic distribution in $\mathcal{D}\mathsf{P}(Q)$. The operator $\mathcal{F}$ allows us to lift the inductive definitions of the Q-families $\mathcal{S}^n_{i+1}$ and $\mathcal{R}^n_{i+1}$ given by Lemma 12, to their counterparts in the level of probability distributions.

▶ **Lemma 15.** *For each $n \in \mathbb{N}$ and $i \in \mathbb{N}$ we have*

$$\vec{\mu_0}\big(\mathcal{S}^n_{i+1}\big) = \mathcal{F}\Big(\vec{\mu_0}\big(\mathcal{S}^n_i\big)\Big) \text{ and } \vec{\mu_0}\big(\mathcal{R}^n_{i+1}\big) = \mathcal{F}\Big(\vec{\mu_0}\big(\mathcal{R}^n_i\big)\Big).$$

[1] That is if $x \leq y$ and $x \in U$ then $y \in U$.

Proof. Take $P \in \mathsf{P}(Q)$ and observe that

$$
\begin{aligned}
\mathcal{F}\Big(\vec{\mu_0}\big(\mathcal{S}_i^n\big)\Big)(P) &\overset{(1)}{=} \frac{1}{|A|} \cdot \sum_{(P_{\mathtt{L}},a,P_{\mathtt{R}})\in\Delta^{-1}(P)} \vec{\mu_0}\big(\mathcal{S}_i^n\big)(P_{\mathtt{L}}) \cdot \vec{\mu_0}\big(\mathcal{S}_i^n\big)(P_{\mathtt{R}})\\
&\overset{(2)}{=} \sum_{(P_{\mathtt{L}},a,P_{\mathtt{R}})\in\Delta^{-1}(P)} \mu_0\big\{t_{\mathtt{L}} \mid \mathcal{S}_i^n[t_{\mathtt{L}}]{=}P_{\mathtt{L}}\big\} \cdot \frac{1}{|A|} \cdot \mu_0\big\{t_{\mathtt{R}} \mid \mathcal{S}_i^n[t_{\mathtt{R}}]{=}P_{\mathtt{R}}\big\}\\
&\overset{(3)}{=} \sum_{(P_{\mathtt{L}},a,P_{\mathtt{R}})\in\Delta^{-1}(P)} \mu_0\big\{t \mid \mathcal{S}_i^n[t{\restriction_{\mathtt{L}}}]{=}P_{\mathtt{L}} \wedge t(\varepsilon){=}a \wedge \mathcal{S}_i^n[t{\restriction_{\mathtt{R}}}]{=}P_{\mathtt{R}}\big\}\\
&\overset{(4)}{=} \mu_0\left(\bigcup_{(P_{\mathtt{L}},a,P_{\mathtt{R}})\in\Delta^{-1}(P)} \big\{t \mid \mathcal{S}_i^n[t{\restriction_{\mathtt{L}}}]{=}P_{\mathtt{L}} \wedge t(\varepsilon){=}a \wedge \mathcal{S}_i^n[t{\restriction_{\mathtt{R}}}]{=}P_{\mathtt{R}}\big\}\right)\\
&\overset{(5)}{=} \mu_0\Big\{t \in \mathrm{Tr}_A \mid \Delta\big(\mathcal{S}_i^n[t{\restriction_{\mathtt{L}}}], t(\varepsilon), \mathcal{S}_i^n[t{\restriction_{\mathtt{R}}}]\big){=}P\Big\}\\
&\overset{(6)}{=} \mu_0\Big\{t \in \mathrm{Tr}_A \mid \mathcal{S}_{i+1}^n[t]{=}P\Big\} \overset{(7)}{=} \vec{\mu_0}\big(\mathcal{S}_{i+1}^n\big)(P),
\end{aligned}
$$

where: (1) is just the definition of $\mathcal{F}\Big(\vec{\mu_0}\big(\mathcal{S}_i^n\big)\Big)$; (2) follows from the definition of $\vec{\mu_0}\big(\mathcal{S}_i^n\big)$; (3) follows from Remark 3 and the independence of $\{t(\varepsilon) = a\}$ from the other events in consideration; (4) follows from the fact that the measured sets are pairwise disjoint; (5) follows simply from the definition of Δ; (6) follows from Lemma 12; and (7) is just the definition of $\vec{\mu_0}\big(\mathcal{S}_{i+1}^n\big)$.

The proof for $\mathcal{R}_{i+1}^n$ follows the same steps, except it uses the $\mathcal{R}_i^n$ variant of Lemma 12. ◀

Now, recall that $Q_{\geq n}$ and $Q_{<n}$ are sets of states of respective priorities. Let the functions $\mathcal{Q}_{<n}, \mathcal{Q}_{\geq n} \colon \mathcal{D}\mathsf{P}(Q) \to \mathcal{D}\mathsf{P}(Q)$ be defined by

$$\mathcal{Q}_{<n}(\beta)(P) \overset{\text{def}}{=} \textstyle\sum_{P'\colon P'\cap Q_{<n}=P} \beta(P'), \tag{4}$$

$$\mathcal{Q}_{\geq n}(\beta)(P) \overset{\text{def}}{=} \textstyle\sum_{P'\colon P'\cup Q_{\geq n}=P} \beta(P'). \tag{5}$$

Again, the formulae guarantee that $\mathcal{Q}_{<n}(\beta)$ and $\mathcal{Q}_{\geq n}(\beta)$ are both probabilistic distributions in $\mathcal{D}\mathsf{P}(Q)$. The following lemma shows the relation between these functions and the limit distributions $\vec{\mu_0}\big(\mathcal{S}_\infty^{n-1}\big)$ and $\vec{\mu_0}\big(\mathcal{R}_\infty^{n-1}\big)$.

▶ **Lemma 16.** *For each $n \in \mathbb{N}$ we have*

$$\mathcal{Q}_{<n}\Big(\vec{\mu_0}\big(\mathcal{S}_\infty^{n-1}\big)\Big) = \vec{\mu_0}\big(\mathcal{R}_0^n\big) \qquad \textit{if } n \textit{ is odd,}$$

$$\mathcal{Q}_{\geq n}\Big(\vec{\mu_0}\big(\mathcal{R}_\infty^{n-1}\big)\Big) = \vec{\mu_0}\big(\mathcal{S}_0^n\big) \qquad \textit{if } n \textit{ is even.}$$

This lemma follows from Lemma 10 in a similar way as Lemma 15 follows from Lemma 12.

Proof. Consider the case of even n and a tree $t \in \mathrm{Tr}_A$. We need to show that

$$\mathcal{Q}_{\geq n}\Big(\vec{\mu_0}\big(\mathcal{R}_\infty^{n-1}\big)\Big) = \vec{\mu_0}\big(\mathcal{S}_0^n\big).$$

Lemma 10 implies that

$$\mathcal{S}_0^n[t] = \big(\mathcal{R}_\infty^{n-1}[t]\big) \cup Q_{\geq n}. \tag{6}$$

Therefore, for each $P \in \mathsf{P}(Q)$ we have

$$\begin{aligned}
\vec{\mu_0}\big(\mathcal{S}^n_0\big)(P) &= \mu_0\big\{t \in \mathrm{Tr}_A \mid \mathcal{S}^n_0[t] = P\big\}\\
&= \mu_0\big\{t \in \mathrm{Tr}_A \mid \big(\mathcal{R}^{n-1}_\infty[t]\big) \cup Q_{\geq n} = P\big\}\\
&= \mu_0\left(\bigcup_{P'\colon P' \cup Q_{\geq n} = P} \big\{t \in \mathrm{Tr}_A \mid \mathcal{R}^{n-1}_\infty[t] = P'\big\}\right)\\
&= \sum_{P'\colon P' \cup Q_{\geq n} = P} \mu_0\big\{t \in \mathrm{Tr}_A \mid \mathcal{R}^{n-1}_\infty[t] = P'\big\}\\
&= \sum_{P'\colon P' \cup Q_{\geq n} = P} \vec{\mu_0}\big(\mathcal{R}^{n-1}_\infty\big)(P')\\
&= \mathcal{Q}_{\geq n}\big(\vec{\mu_0}\big(\mathcal{R}^{n-1}_\infty\big)\big)(P)
\end{aligned}$$

The case of odd n is entirely analogous. ◀

The two above lemmata express the properties of the operators $\mathcal{F}$, $\mathcal{Q}_{<n}$, and $\mathcal{Q}_{\geq n}$ as depicted on Figure 1.

6 Limit distributions $\vec{\mu_0}\big(\mathcal{S}^n_\infty\big)$ and $\vec{\mu_0}\big(\mathcal{R}^n_\infty\big)$

In this section we show how to represent the distributions $\vec{\mu_0}\big(\mathcal{S}^n_\infty\big)$ and $\vec{\mu_0}\big(\mathcal{R}^n_\infty\big)$ as fixed points. We begin by proving that these distributions are limits in $\mathbb{R}^{\mathsf{P}(Q)}$ of the sequences of vectors $\big(\vec{\mu_0}\big(\mathcal{S}^n_i\big)\big)_{i\in\mathbb{N}}$ and $\big(\vec{\mu_0}\big(\mathcal{R}^n_i\big)\big)_{i\in\mathbb{N}}$ respectively. This is a consequence of Lemmata 7 and 9.

▶ **Lemma 17.** *For each $n \in \mathbb{N}$ and $P \in \mathsf{P}(Q)$ we have*

$$\lim_{i\to\infty} \vec{\mu_0}\big(\mathcal{S}^n_i\big)(P) = \vec{\mu_0}\big(\mathcal{S}^n_\infty\big)(P) \text{ and } \lim_{i\to\infty} \vec{\mu_0}\big(\mathcal{R}^n_i\big)(P) = \vec{\mu_0}\big(\mathcal{R}^n_\infty\big)(P).$$

Proof. We consider case of $\vec{\mu_0}\big(\mathcal{S}^n_\infty\big)$, the case of $\vec{\mu_0}\big(\mathcal{R}^n_\infty\big)(P)$ is entirely dual. First, we show that the respective limits agree when taking sums over any upward closed family $U \subseteq \mathsf{P}(Q)$, see (7) below. For $i \in \mathbb{N}$ let $X_i = \bigcup_{P'\in U}\{t \in \mathrm{Tr}_A \mid \mathcal{S}^n_i[t] = P'\}$ and $X_\infty = \bigcup_{P'\in U}\{t \in \mathrm{Tr}_A \mid \mathcal{S}^n_\infty[t] = P'\}$. Lemma 7 together with the fact that U is upward-closed imply that $X_0 \supseteq X_1 \supseteq \ldots \supseteq X_\infty$. Lemma 9 and finiteness of Q imply that for every tree t there exists an index J such that $\mathcal{S}^n_J[t] \subseteq \mathcal{S}^n_\infty[t]$. Therefore, $\bigcap_{i\in\mathbb{N}} X_i = X_\infty$. By continuity of the measure μ_0 we get that $\lim_{i\to\infty} \mu_0(X_i) = \mu_0(X_\infty)$. This means that

$$\lim_{i\to\infty} \sum_{P'\in U} \vec{\mu_0}\big(\mathcal{S}^n_i\big)(P') = \lim_{i\to\infty} \mu_0(X_i) = \mu_0(X_\infty) = \sum_{P'\in U} \vec{\mu_0}\big(\mathcal{S}^n_\infty\big)(P'). \tag{7}$$

Clearly, $\{P\} = \{P' \in \mathsf{P}(Q) \mid P' \supseteq P\} \setminus \{P' \in \mathsf{P}(Q) \mid P' \supsetneq P\}$ with both these families upward closed. Therefore, we can apply (7) twice and obtain the desired equation. ◀

The monotonicity of Δ, see Lemma 11, implies the following lemma.

▶ **Lemma 18.** *The operator $\mathcal{F}\colon \mathcal{D}\mathsf{P}(Q) \to \mathcal{D}\mathsf{P}(Q)$, see Equation (3), is monotone in $\preceq$.*

Proof. We need to prove that $\mathcal{F}$ is monotone w.r.t. the order $\preceq$. Thus, for every $\alpha \preceq \beta \in \mathcal{D}\mathsf{P}(Q)$ and an upward-closed family $U \subseteq \mathsf{P}(Q)$ we should have $\sum_{P\in U} \mathcal{F}(\alpha)(P) \leq \sum_{P\in U} \mathcal{F}(\beta)(P)$.

After multiplying by $\frac{1}{|A|}$ and splitting the sum over separate letters $a \in A$ (see the definition of $\mathcal{F}$, cf. (3)), it is enough to show that for each $a \in A$ and $O_a \stackrel{\text{def}}{=} \{(P_{\mathtt{L}}, P_{\mathtt{R}}) \mid \Delta(P_{\mathtt{L}}, a, P_{\mathtt{R}}) \in U\}$ we have

$$\sum_{(P_{\mathtt{L}},P_{\mathtt{R}})\in O_a} \alpha(P_{\mathtt{L}}) \cdot \alpha(P_{\mathtt{R}}) \leq \sum_{(P_{\mathtt{L}},P_{\mathtt{R}})\in O_a} \beta(P_{\mathtt{L}}) \cdot \beta(P_{\mathtt{R}}).$$

Now, by monotonicity of Δ (see Lemma 11) and the fact that U is upward-closed, we know that if $P_{\mathtt{L}} \subseteq P'_{\mathtt{L}}$, $P_{\mathtt{R}} \subseteq P'_{\mathtt{R}}$, and $(P_{\mathtt{L}}, P_{\mathtt{R}}) \in O_a$ then also $(P'_{\mathtt{L}}, P'_{\mathtt{R}}) \in O_a$. By $P_{\mathtt{L}}^{-1} \cdot O_a$ and $O_a \cdot P_{\mathtt{R}}^{-1}$ we will denote the sections $\{P_{\mathtt{R}} \mid (P_{\mathtt{L}}, P_{\mathtt{R}}) \in O_a\}$ and $\{P_{\mathtt{L}} \mid (P_{\mathtt{L}}, P_{\mathtt{R}}) \in O_a\}$ respectively. Notice that both of them are upward-closed. Thus, using the assumption that $\alpha \preceq \beta$ twice, we obtain

$$\begin{aligned}
\sum_{(P_{\mathtt{L}},P_{\mathtt{R}})\in O_a} \alpha(P_{\mathtt{L}}) \cdot \alpha(P_{\mathtt{R}}) &= \sum_{P_{\mathtt{L}}\in\mathsf{P}(Q)} \alpha(P_{\mathtt{L}}) \cdot \left(\sum_{P_{\mathtt{R}}\in P_{\mathtt{L}}^{-1}\cdot O_a} \alpha(P_{\mathtt{R}}) \right) \\
&\leq \sum_{P_{\mathtt{L}}\in\mathsf{P}(Q)} \alpha(P_{\mathtt{L}}) \cdot \left(\sum_{P_{\mathtt{R}}\in P_{\mathtt{L}}^{-1}\cdot O_a} \beta(P_{\mathtt{R}}) \right) \\
&= \sum_{(P_{\mathtt{L}},P_{\mathtt{R}})\in O_a} \alpha(P_{\mathtt{L}}) \cdot \beta(P_{\mathtt{R}}) = \sum_{(P_{\mathtt{L}},P_{\mathtt{R}})\in O_a} \beta(P_{\mathtt{R}}) \cdot \alpha(P_{\mathtt{L}}) \\
&= \sum_{P_{\mathtt{R}}\in\mathsf{P}(Q)} \beta(P_{\mathtt{R}}) \cdot \left(\sum_{P_{\mathtt{L}}\in O_a\cdot P_{\mathtt{R}}^{-1}} \alpha(P_{\mathtt{L}}) \right) \\
&\leq \sum_{P_{\mathtt{R}}\in\mathsf{P}(Q)} \beta(P_{\mathtt{R}}) \cdot \left(\sum_{P_{\mathtt{L}}\in O_a\cdot P_{\mathtt{R}}^{-1}} \beta(P_{\mathtt{L}}) \right) \\
&= \sum_{(P_{\mathtt{L}},P_{\mathtt{R}})\in O_a} \beta(P_{\mathtt{R}}) \cdot \beta(P_{\mathtt{L}}) = \sum_{(P_{\mathtt{L}},P_{\mathtt{R}})\in O_a} \beta(P_{\mathtt{L}}) \cdot \beta(P_{\mathtt{R}}).
\end{aligned}$$

◀

With the two lemmata above, we are ready to conclude this section: we characterise the distributions $\vec{\mu_0}\big(\mathcal{S}^n_\infty\big)$ and $\vec{\mu_0}\big(\mathcal{R}^n_\infty\big)$, see Figure 1, by a specialised variant of the Knaster-Tarski fixed point theorem.

▶ **Proposition 19.** *For each $n \in \mathbb{N}$ the distribution $\vec{\mu_0}\big(\mathcal{S}^n_\infty\big)$ is the $\preceq$-greatest distribution β satisfying $\beta = \mathcal{F}(\beta)$ and $\beta \preceq \vec{\mu_0}\big(\mathcal{S}^n_0\big)$. Similarly, $\vec{\mu_0}\big(\mathcal{R}^n_\infty\big)$ is the $\preceq$-least distribution β satisfying $\beta = \mathcal{F}(\beta)$ and $\beta \succeq \vec{\mu_0}\big(\mathcal{R}^n_0\big)$.*

Proof. Consider the case of $\vec{\mu_0}\big(\mathcal{S}^n_\infty\big)$. Observe that $\mathcal{F}$ is continuous in $\mathbb{R}^{\mathsf{P}(Q)}$. Indeed, it is given by a vector of quadratic polynomials from $\mathbb{R}^{\mathsf{P}(Q)}$ to $\mathbb{R}^{\mathsf{P}(Q)}$. Now, take $P \in \mathsf{P}(Q)$ and observe that

$$\begin{aligned}
\vec{\mu_0}\big(\mathcal{S}^n_\infty\big)(P) = \lim_{i\to\infty} \vec{\mu_0}\big(\mathcal{S}^n_i\big)(P) = \lim_{i\to\infty} \mathcal{F}\big(\vec{\mu_0}\big(\mathcal{S}^n_i\big)\big)(P) = \\
\mathcal{F}\Big(\lim_{i\to\infty} \vec{\mu_0}\big(\mathcal{S}^n_i\big)(P)\Big) = \mathcal{F}\Big(\vec{\mu_0}\big(\mathcal{S}^n_\infty\big)(P)\Big)
\end{aligned}$$

where the first equality follows from Lemma 17; the second from Lemma 15; the third from continuity of $\mathcal{F}$; and the last from Lemma 17, again. Therefore, $\beta = \vec{\mu_0}\big(\mathcal{S}^n_\infty\big)$ satisfies $\beta = \mathcal{F}(\beta)$. Moreover, Lemmata 7 and 14 imply that $\beta \preceq \vec{\mu_0}\big(\mathcal{S}^n_0\big)$.

Consider now any distribution $\beta \in \mathcal{D}\mathsf{P}(Q)$ such that $\beta = \mathcal{F}(\beta)$ and $\beta \preceq \vec{\mu_0}\big(\mathcal{S}^n_0\big)$. We need to prove that $\beta \preceq \vec{\mu_0}\big(\mathcal{S}^n_\infty\big)$. Lemma 18 states that $\mathcal{F}$ is monotone. Therefore, by inductively applying Lemma 15 for $i = 0, \ldots,$ we infer that

$$\beta = \mathcal{F}(\beta) \le \mathcal{F}\Big(\vec{\mu_0}\big(\mathcal{S}^n_i\big)\Big) = \vec{\mu_0}\big(\mathcal{S}^n_{i+1}\big).$$

Take any upward-closed family $U \subseteq \mathsf{P}(Q)$. The above inequality implies that for each $i \in \mathbb{N}$ we have $\sum_{P\in U} \beta(P) \le \sum_{P\in U} \vec{\mu_0}\big(\mathcal{S}^n_i\big)(P)$. By taking the limit as in Lemma 17 we obtain that $\sum_{P\in U} \beta(P) \le \sum_{P\in U} \vec{\mu_0}\big(\mathcal{S}^n_\infty\big)(P)$. This implies that $\beta \preceq \vec{\mu_0}\big(\mathcal{S}^n_\infty\big)$.

The case of $\vec{\mu_0}\big(\mathcal{R}^n_\infty\big)$ is similar, we utilise the opposite monotonicity $\beta \succeq \vec{\mu_0}\big(\mathcal{R}^n_{i+1}\big)$. ◀

7 Computing measures

In this section, we conclude our solution to Problem 1 for weak alternating automata. This is achieved by a reduction to the first-order theory of the real numbers $\mathcal{R} = \langle \mathbb{R}, 0, 1, +, \cdot\rangle$. The theory is famously decidable thanks to Tarski-Seidenberg theorem, see e.g. [25].

Throughout this section, we assume that the reader is familiar with the syntax and semantics of first-order logic. We say that a formula $\varphi(x_1, \ldots, x_k)$ *represents* a relation $r \subseteq \mathbb{R}^k$ if it holds in $\mathbb{R}$ according to an evaluation v of the free variables $x_1, \ldots, x_n$, precisely when the tuple $\langle v(x_1), \ldots, v(x_k)\rangle$ belongs to r. For example, the formula $\exists z.\ x + (z{\cdot}z) = y$ represents the standard ordering $\le$ on real numbers. A formula represents a number $a \in \mathbb{R}$ if it represents the singleton $\{a\}$; for example the formula $x{\cdot}x = 1{+}1 \land \exists z.\ x = z{\cdot}z$, represents the number $\sqrt{2}$.

▶ **Theorem 20.** *Given a weak alternating automaton $\mathcal{A}$ one can compute a formula $\psi_{\mathcal{A}}(x)$ that represents the number $\mu_0\big(\mathrm{L}(\mathcal{A})\big)$. Moreover, the formula is in a prenex normal form, its size is exponential in the size of $\mathcal{A}$, and the quantifier alternation of $\psi_{\mathcal{A}}(x)$ is constant.*

Proof. Fix a weak alternating automaton $\mathcal{A} = \langle A, Q, q_{\mathrm{I}}, \delta, \Omega\rangle$. Let $N > \Omega(q_{\mathrm{I}})$ be an even number (either $\Omega(q_{\mathrm{I}}){+}1$ or $\Omega(q_{\mathrm{I}}){+}2$). Fix an enumeration $\{P_1, \ldots, P_K\}$ of $\mathsf{P}(Q)$ with $K = 2^{|Q|}$. We will identify a distribution $\alpha \in \mathcal{D}\mathsf{P}(Q)$ with its representation $\alpha = (a_1, \ldots, a_K) \in \mathbb{R}^K$ as a vector of real numbers. Following this identification, $\alpha(P_k)$ stands for a_k. Clearly the properties that $\mathcal{F}(\alpha) = \beta$, $\mathcal{Q}_{<n}(\alpha) = \beta$, and $\mathcal{Q}_{\ge n}(\alpha) = \beta$ are definable by quantifier free formulae of size polynomial in K.

The following formula defines the fact that $\alpha \in \mathcal{D}\mathsf{P}(Q)$.

$$\mathrm{dist}(\alpha) \equiv \sum_{k=1}^{K} \alpha(P_k) = 1 \ \land\ \bigwedge_{k=1}^{K} 0 \le \alpha(P_k) \le 1$$

Analogously to our representation of distributions, every subset $U \subseteq \mathsf{P}(Q)$ can be represented by its indicator: a vector of numbers $\iota = (i_1, \ldots, i_K)$ such that $\iota(P)$ is either 0 (if $P \notin U$) or 1 (if $P \in U$). Note that if U is upward closed then whenever $P \subseteq P'$ and $\iota(P) = 1$ then $\iota(P') = 1$. The following formula defines the fact that ι represents an upward-closed set.

$$\mathrm{upward}(\iota) \equiv \bigwedge_{k=1}^{K} \big(\iota(P_k){=}0 \lor \iota(P_k){=}1\big) \land \bigwedge_{P\subseteq P'} \iota(P){=}1 \to \iota(P'){=}1.$$

Thus, to check if $\alpha \preceq \beta$ one can use the following formula (see Claim 21 below)

$$\mathrm{minor}(\alpha, \beta, \iota) \equiv \sum_{k=1}^{K} \alpha(P_k) \cdot \iota(P_k) \le \sum_{k=1}^{K} \beta(P_k) \cdot \iota(P_k).$$

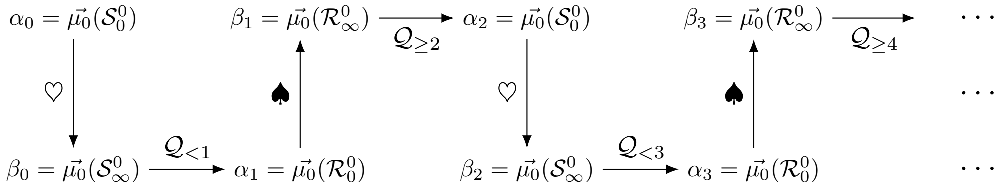

Figure 2 A diagram of the distributions α_n and β_n in the formula $\psi_{\mathcal{A}}(x)$, cf. Figure 1. The symbol ♡ represents applications of Proposition 19 in the case of $\mathcal{S}^n_0$ and $\mathcal{S}^n_\infty$; while ♠ corresponds to the dual case of $\mathcal{R}^n_0$ and $\mathcal{R}^n_\infty$.

Notice that all the above formulae: $\mathrm{dist}(\alpha)$, $\mathrm{upward}(\iota)$, and $\mathrm{minor}(\alpha,\beta,\iota)$ are quantifier free: the $\bigwedge$ there are just explicitly written as finite conjunctions. Therefore, these formulae can be used to relativise quantifiers in a prenex normal form of a formula: for instance we write $\forall\alpha\colon \mathrm{dist}(\alpha).\ \exists\beta\colon \mathrm{dist}(\beta).\ \psi(\alpha,\beta)$ to denote $\forall\alpha.\ \exists\beta.\ \mathrm{dist}(\alpha) \rightarrow \big(\mathrm{dist}(\beta) \wedge \psi(\alpha,\beta)\big)$.

▷ Claim 21. Given two distributions α and β, we have $\alpha \preceq \beta$ if and only if

$$\forall\iota\colon \mathrm{upward}(\iota).\ \mathrm{minor}(\alpha,\beta,\iota).$$

The formula $\psi_{\mathcal{A}}(x)$ is indented to specify the distributions $(\alpha_n,\beta_n)_{n=0,\ldots,N}$ in a way depicted on Figure 2. The value $\vec{\mu_0}\big(\mathcal{S}^0_0(P)\big)$ is 1 if $P = Q$ and 0 otherwise, see Lemma 10. Proposition 19 allows us to define $\vec{\mu_0}\big(\mathcal{S}^n_\infty\big)$ (resp. $\vec{\mu_0}\big(\mathcal{R}^n_\infty\big)$) using $\vec{\mu_0}\big(\mathcal{S}^n_0\big)$ (resp. $\vec{\mu_0}\big(\mathcal{R}^n_0\big)$) as specific fixed points of the operation $\mathcal{F}$. Finally, Lemma 16 allows us to define $\vec{\mu_0}\big(\mathcal{R}^n_0\big)$ using $\mathcal{Q}_{<n}$, and $\vec{\mu_0}\big(\mathcal{S}^n_0\big)$ using $\mathcal{Q}_{\geq n}$. The value of x is related to those distributions based on Lemma 10 which implies that $\mu_0\big(\mathrm{L}(\mathcal{A})\big) = \sum_{P\colon q_{\mathrm{I}}\in P\in\mathsf{P}(Q)} \vec{\mu_0}\big(\mathcal{S}^N_0\big)(P)$.

The following equation defines the formula $\psi_{\mathcal{A}}(x)$.

$$\psi_{\mathcal{A}}(x) \equiv \exists\alpha_0,\beta_0\colon \mathrm{dist}(\alpha_0), \mathrm{dist}(\beta_0), \beta_0{=}\mathcal{F}(\beta_0).$$

$$\vdots \tag{8}$$

$$\exists\alpha_N,\beta_N\colon \mathrm{dist}(\alpha_N), \mathrm{dist}(\beta_N), \beta_N{=}\mathcal{F}(\beta_N).$$

$$\forall\theta\colon \mathrm{dist}(\theta), \theta{=}\mathcal{F}(\theta). \tag{9}$$

$$\exists\iota_0\colon \mathrm{upward}(\iota_0).$$

$$\vdots \tag{10}$$

$$\exists\iota_N\colon \mathrm{upward}(\iota_N).$$

$$\forall\gamma_0\colon \mathrm{upward}(\gamma_0).$$

$$\vdots \tag{11}$$

$$\forall\gamma_N\colon \mathrm{upward}(\gamma_N).$$

$$\left(\alpha_0(Q) = 1 \wedge \bigwedge_{P\neq Q} \alpha_0(P) = 0\right) \wedge \tag{12}$$

$$\left(\bigwedge_{n=1}^{N} [n \text{ is odd}] \rightarrow \alpha_n = \mathcal{Q}_{<n}(\beta_{n-1})\right) \wedge \tag{13}$$

$$\left(\bigwedge_{n=1}^{N} [n \text{ is even}] \rightarrow \alpha_n = \mathcal{Q}_{\geq n}(\beta_{n-1})\right) \wedge \tag{14}$$

$$\left(\bigwedge_{n=0}^{N}[n\text{ is odd}] \to \mathrm{minor}(\alpha_n,\beta_n,\gamma_n)\right)\ \wedge \tag{15}$$

$$\left(\bigwedge_{n=0}^{N}[n\text{ is even}] \to \mathrm{minor}(\beta_n,\alpha_n,\gamma_n)\right)\ \wedge \tag{16}$$

$$\left(\bigwedge_{n=0}^{N}[n\text{ is odd}] \to \Big(\neg\mathrm{minor}(\alpha_n,\theta,\iota_n)\vee \mathrm{minor}(\beta_n,\theta,\gamma_n)\Big)\right)\ \wedge \tag{17}$$

$$\left(\bigwedge_{n=0}^{N}[n\text{ is even}] \to \Big(\neg\mathrm{minor}(\theta,\alpha_n,\iota_n)\vee \mathrm{minor}(\theta,\beta_n,\gamma_n)\Big)\right)\ \wedge \tag{18}$$

$$\left(\sum_{P\ni q_{\mathrm{I}}}\alpha_N(P)=x\right) \tag{19}$$

Observe that the size of this formula is polynomial in K and N (in fact it is $\mathcal{O}(N\cdot K^2)$), i.e. exponential in the size of the automaton $\mathcal{A}$. Moreover, the formula is in prenex normal form and its quantifier alternation is 4 (the sub-formulae that involve $\bigwedge$ are written explicitly as conjunctions).

We begin by proving soundness of the formula: we assume that $\psi_{\mathcal{A}}(x)$ holds and show that $x=\mu_0\big(\mathrm{L}(\mathcal{A})\big)$. Consider a sequence of distributions $(\alpha_n,\beta_n)_{n=0,\ldots,N}$ witnessing (8). The following two lemmata prove inductively that for $n=0,\ldots,N$ we have

$$\begin{aligned} &\alpha_n=\vec{\mu_0}(\mathcal{S}_0^n)\text{ and }\beta_n=\vec{\mu_0}(\mathcal{S}_\infty^n) && \text{for even } n,\\ &\alpha_n=\vec{\mu_0}(\mathcal{R}_0^n)\text{ and }\beta_n=\vec{\mu_0}(\mathcal{R}_\infty^n) && \text{for odd } n. \end{aligned} \tag{20}$$

▶ **Lemma 22.** *Using the above notations and the assumption that $\psi_{\mathcal{A}}(x)$ holds:*

for even n, if $\alpha_n=\vec{\mu_0}(\mathcal{S}_0^n)$ then $\beta_n=\vec{\mu_0}(\mathcal{S}_\infty^n)$,
for odd n, if $\alpha_n=\vec{\mu_0}(\mathcal{R}_0^n)$ then $\beta_n=\vec{\mu_0}(\mathcal{R}_\infty^n)$.

Proof. Both claims follow from Proposition 19. Take n odd and assume that $\alpha_n=\vec{\mu_0}(\mathcal{R}_0^n)$. We know that $\beta_n=\mathcal{F}(\beta_n)$ by (8). Moreover, by Claim 21, the arbitrary choice of γ_n, and (15) we know that $\alpha_n\preceq\beta_n$. It is enough to prove that if θ is any distribution satisfying $\alpha_n\preceq\theta$ and $\theta=\mathcal{F}(\theta)$ then $\beta_n\preceq\theta$.

Assume contrarily that θ is a distribution such that $\alpha_n\preceq\theta$ and $\theta=\mathcal{F}(\theta)$ but $\beta_n\not\preceq\theta$. We know that θ must satisfy the sub-formula in (9). Take the upward closed sets $(\iota_\ell)_{\ell=0,\ldots,N}$ given by (10). Now let $(\gamma_\ell)_{\ell=0,\ldots,N}$ be any sequence of upward closed sets such that γ_n witnesses the fact that $\beta_n\not\preceq\theta$, i.e. $\neg\mathrm{minor}(\beta_n,\theta,\gamma_n)$ holds. But this is a contradiction with (17) because $\mathrm{minor}(\alpha_n,\theta,\iota_n)$ is true as $\alpha_n\preceq\theta$ and $\mathrm{minor}(\beta_n,\theta,\gamma_n)$ is false.

The case of even n is analogous. ◀

▶ **Lemma 23.** *Using the above notations and the assumption that $\psi_{\mathcal{A}}(x)$ holds:*

$$\alpha_n=\vec{\mu_0}(\mathcal{S}_0^n)\textit{ for even } n\quad\textit{and}\quad\alpha_n=\vec{\mu_0}(\mathcal{R}_0^n)\textit{ for odd } n.$$

Proof. The proof is inductive in n. First, $\alpha_0=\vec{\mu_0}(\mathcal{S}_0^n)$ because of (12) and the statement for $n=0$ in Lemma 10 (we can take $\theta=\beta_0$ and $\gamma_\ell=\iota_\ell$ for $\ell=0,\ldots,N$ to check that Condition (12) holds).

Now assume that the above conditions are true for $n-1$ for some $n\in\{1,\ldots,N\}$. Again, by the symmetry we assume that n is odd, i.e. $\alpha_{n-1}=\vec{\mu_0}(\mathcal{S}_0^{n-1})$. By Lemma 22 we know that $\beta_{n-1}=\vec{\mu_0}(\mathcal{S}_\infty^{n-1})$. Condition (13) says that $\alpha_n=\mathcal{Q}_{<n}(\beta_{n-1})=\mathcal{Q}_{<n}\big(\vec{\mu_0}(\mathcal{S}_\infty^{n-1})\big)$. Now Lemma 16 implies that $\mathcal{Q}_{<n}\big(\vec{\mu_0}(\mathcal{S}_\infty^{n-1})\big)=\vec{\mu_0}\big(\mathcal{R}_0^n\big)$ and the induction step is complete. ◀

Equation (20) together with Condition (19), imply that $x = \mu_0\{t \in \mathrm{Tr}_A \mid q_{\mathrm{I}} \in \mathcal{S}_0^N[t]\}$. Since $N > \Omega(q_{\mathrm{I}})$ is even, Lemma 10 implies that $\mathcal{S}_0^N(q_{\mathrm{I}}) = \mathrm{L}(\mathcal{A}, q_{\mathrm{I}})$ and therefore, $q_{\mathrm{I}} \in \mathcal{S}_0^N[t]$ if and only if $t \in \mathrm{L}(\mathcal{A})$. This guarantees that $x = \mu_0\big(\mathrm{L}(\mathcal{A})\big)$.

We will now prove completeness of the formula: if $x = \mu_0\big(\mathrm{L}(\mathcal{A})\big)$ then $\psi_{\mathcal{A}}(x)$ holds. Choose the distributions $(\alpha_n, \beta_n)_{n=0,\ldots,N}$ in (8) as in (20). We will show that then the rest of the formula holds. Consider any distribution θ. For each $n = 0, \ldots, N$ let ι_n be an upward-closed set witnessing that $\alpha_n \not\preceq \theta$ for n odd (resp. $\theta \not\preceq \alpha_n$ for n even); or any upward closed set if the respective inequality holds.

Take any $(\gamma_n)_{n=0,\ldots,N}$ that are upward closed. We need to check that the sub-formula starting in (12) holds. Conditions (12) — (16) and (19) hold by the same lemmata as mentioned in the previous section. To check Conditions (17) and (18) one again invokes Proposition 19: either ι_n witnesses that $\alpha_n \not\preceq \theta$ (resp. $\theta \not\preceq \alpha_n$) or, if ι_n was chosen arbitrarily, then Proposition 19 implies that also the respective inequality with β_n holds. ◀

8 Branching processes

For the sake of simplicity we define only binary branching processes, the case of a fixed higher arity can be solved analogously. A *branching process* is a tuple $\mathcal{P} = \langle A, \tau, \alpha_{\mathrm{I}}\rangle$ where A is a finite alphabet; $\tau\colon A \to \mathcal{D}A^2$ a *branching function* that assigns a probability distribution over A^2 to every letter in A; and $\alpha_{\mathrm{I}} \in \mathcal{D}A$ an *initial distribution*. We assume that all probabilities occurring in these distributions are rational. By the *size* of $\mathcal{P}$ we understand the size of its binary representation.

A branching process $\mathcal{P}$ can be seen as a generator of random trees: it defines a complete Borel measure $\mu_{\mathcal{P}}$ over the set of infinite trees in the following way. Let $f\colon \mathrm{dom}(f) \to A$ be a complete finite tree of depth $d \geq 0$ i.e. $\mathrm{dom}(f) = \{u \in \{\mathtt{L},\mathtt{R}\}^* \mid |u| \leq d\} = \{\mathtt{L},\mathtt{R}\}^{<d+1}$. Then the measure $\mu_{\mathcal{P}}$ of the basic set U_f, see Section 2, is defined by

$$\mu_{\mathcal{P}}(U_f) \stackrel{\text{def}}{=} \alpha_{\mathrm{I}}(f(\varepsilon)) \cdot \prod_{u \in \{\mathtt{L},\mathtt{R}\}^{<d}} \tau(f(u))\big(f(u\mathtt{L}), f(u\mathtt{R})\big). \tag{21}$$

Now, $\mu_{\mathcal{P}}$ can be extended in a standard way to a complete Borel measure on the set of all infinite trees Tr_A. Intuitively, a tree $t \in \mathrm{Tr}_A$ that is chosen according to $\mu_{\mathcal{P}}$ is generated in a top-down fashion: the root label $t(\varepsilon)$ is chosen according to the initial distribution α_{I}; and the labels of the children $u\mathtt{L}$ and $u\mathtt{R}$ of a node u are chosen according to the distribution $\tau(t(u)) \in \mathcal{D}A^2$ defined for the label of their parent u.

Observe that the uniform measure μ_0 over trees Tr_A equals the measure $\mu_{\mathcal{P}_0}$ defined by the branching process $\mathcal{P}_0 = \langle A, \tau_0, \alpha_0\rangle$, where $\alpha_0(a) = |A|^{-1}$ and $\tau_0(a)(a_{\mathtt{L}}, a_{\mathtt{R}}) = |A|^{-2}$ for each $a, a_{\mathtt{L}}, a_{\mathtt{R}} \in A$.

▶ **Theorem 24.** *Given a weak alternating automaton $\mathcal{A}$ and a branching process $\mathcal{P}$ one can compute a formula $\psi_{\mathcal{A},\mathcal{P}}(x)$ that represents the number $\mu_{\mathcal{P}}\big(\mathrm{L}(\mathcal{A})\big)$. Moreover, the formula is in a prenex normal form; its size is exponential in the size of $\mathcal{A}$ and polynomial in the size of $\mathcal{P}$; and the quantifier alternation of $\psi_{\mathcal{A},\mathcal{P}}$ is constant.*

If one does not care about the complexity, the above result can be obtained directly, by interpreting the branching process $\mathcal{P}$ in an automaton $\mathcal{A}$. More precisely, there exists an algorithm that, given a weak alternating automaton $\mathcal{A}$ and a branching process $\mathcal{P}$, computes another weak alternating automaton $\mathcal{A}_{\mathcal{P}}$ such that

$$\mu_{\mathcal{P}}\big(\mathrm{L}(\mathcal{A})\big) = \mu_0\big(\mathrm{L}(\mathcal{A}_{\mathcal{P}})\big).$$

Therefore, the decidability part of Theorem 24 follows directly from Theorem 20. A construction of $\mathcal{A}_\mathcal{P}$ is given in Subsection 8.1. Another advantage of the construction given there is that it deals explicitly with branching processes of arbitrary branching (possibly non-binary). However, it is possible to provide a direct way of constructing the formula $\psi_{\mathcal{A},\mathcal{P}}$ with the size of the formula polynomial in the size of $\mathcal{P}$, see Subjection 8.2.

8.1 Encoding branching processes in automata

This section shows how to use the expressive power of weak MSO to simulate branching processes within the uniform measure.

An *ℓ-branching tree* over an alphabet A is a function $t\colon \{\mathtt{D}_1,\ldots,\mathtt{D}_\ell\}^* \to A$, where $\mathtt{D}_1,\ldots,\mathtt{D}_\ell$ are ℓ distinct symbols (we assume that $\mathtt{L} = \mathtt{D}_1$ and $\mathtt{R} = \mathtt{D}_2$). The set of all such trees is denoted $\mathrm{Tr}_A^{(\ell)}$.

Similarly, an *ℓ-branching process* $\mathcal{P} = \langle A, \tau, \alpha_\mathrm{I}\rangle$ is defined analogously to a branching process, except that a branching function $\tau\colon A \to \mathcal{D}A^\ell$ randomly produces ℓ-tuples of letters. This implies that the measure $\mu_\mathcal{P}$ is a Borel measure over the set of ℓ-branching trees $\mathrm{Tr}_A^{(\ell)}$.

An *ℓ-branching* alternating automaton $\mathcal{A}$ is again analogous to a standard alternating automaton but the atoms (d, q') in the transition formulae satisfy $d \in \{\mathtt{D}_1,\ldots,\mathtt{D}_\ell\}$. If t is an ℓ-branching tree and $\mathcal{A}$ is an ℓ-branching automaton, then the game $\mathcal{G}(t,p)$ is defined analogously as in Section 2. Thus, the language $\mathrm{L}(\mathcal{A})$ is a subset of $\mathrm{Tr}_A^{(\ell)}$.

According to the above definitions, standard trees, branching processes, and automata, as defined in the main body of this article, are 2-branching.

▶ **Proposition 25.** *Let $\mathcal{A}$ be a weak ℓ-branching alternating automaton over an alphabet A and $\mathcal{P}$ be a ℓ-branching process. Let A_0 be any alphabet with at least two symbols. Then, one can construct a weak 2-branching alternating automaton $\mathcal{A}_\mathcal{P}$ over the alphabet A_0 such that $\mu_\mathcal{P}\big(\mathrm{L}(\mathcal{A})\big) = \mu_0\big(\mathrm{L}(\mathcal{A}_\mathcal{P})\big)$, where μ_0 is the uniform measure over 2-branching trees Tr_{A_0}.*

Notice that for $\ell = 2$ this reduction is made redundant by the results of Subsection 8.2, which allows us to directly compute $\mu_\mathcal{P}\big(\mathrm{L}(\mathcal{A})\big)$. Moreover, the construction provided there has better complexity: the obtained formula $\psi_{\mathcal{A},\mathcal{P}}$ is only polynomial in the size of $\mathcal{P}$. However, we provide the present reduction because it shows that the class of languages recognisable by weak alternating automata is robust. In particular, if one does not care about the size of the respective formulae, then Theorem 24 can be obtained via the above reduction directly from Theorem 20. Also, this is the only place in the article when we explicitly deal with branching processes of higher branching than 2.

We start with an encoding of rational numbers.

▶ **Lemma 26.** *Let X be a finite set, A_0 any alphabet with at least two symbols, and $\alpha \in \mathcal{D}X$ a probabilistic distribution with rational values. Then there exists a weak alternating automaton $\mathcal{A}_\alpha$ over the alphabet A_0 with a set of states Q_α and a function $j\colon X \to Q_\alpha$ such that:*

- *for $x \neq x' \in X$ the languages $\mathrm{L}(\mathcal{A}_\alpha, j(x))$ and $\mathrm{L}(\mathcal{A}_\alpha, j(x'))$ are disjoint;*
- *the union $\bigcup_{x\in X} \mathrm{L}(\mathcal{A}_\alpha, j(x))$ is the set of all trees $\mathrm{Tr}_{\{0,1\}}$;*
- *for every $x \in X$ the measure $\mu_0\big(\mathrm{L}(\mathcal{A}_\alpha, j(x))\big)$ equals $\alpha(x)$.*

Proof. Without loss of generality we can assume that $A_0 = \{0,\ldots,|A_0|-1\}$. Assume that $X = \{x_1,\ldots,x_K\}$. Fix rational numbers $r_k \stackrel{\text{def}}{=} \sum_{k'\leq k} \alpha(x_{k'})$ for $k = 0,\ldots,K$. We know that $r_0 = 0$ and $r_K = 1$. For each $k = 0,\ldots,K$ let e_k be the M-ary expansion of r_k, i.e. $e_k \in A_0^\omega$ is a word such that $r_k = 0.e_k$. Since each of the numbers r_k is rational, the words e_k are ultimately periodic, i.e. of the form $u \cdot v \cdot v \cdot v \cdots$

Each tree $t \in \mathrm{Tr}_{A_0}$ induces a real number $r(t) \in [0,1]$ that is obtained by reading the left-most branch of t and treating it as an $|A_0|$-ary expansion of $r(t)$.

Let $e, e' \in A_0^\omega$ be two expansions of rational numbers with $0.e < 0.e'$. It is now standard to construct a weak deterministic automaton $\mathcal{A}_{e,e'}$ with an initial state $q_{e,e'}$ that accepts a tree $t \in \mathrm{Tr}_{A_0}$ if and only if $0.e \leq r(t) < 0.e'$.

Now, to obtain the automaton $\mathcal{A}_\alpha$ it is enough to take the disjoint union of the automata $\mathcal{A}_{e_{k-1},e_k}$ for $k = 1, \ldots, K$ and define $j(x_k) = q_{e_{k-1},e_k}$. ◀

We now move to the proof of Proposition 25. Take a weak ℓ-branching alternating automaton over an alphabet A and an ℓ-branching process $\mathcal{P}$ over the same alphabet. For the sake of simplicity assume that the initial distribution α_{I} of $\mathcal{P}$ is concentrated in a single letter $a_{\mathrm{I}} \in A$.

The above construction will be used to simulate the random choice represented by the distributions $\tau(a) \in \mathcal{D}A^\ell$. The automaton $\mathcal{A}_\mathcal{P}$ is defined as a disjoint union of the automata $\mathcal{A}_{\tau(a)}$ for each $a \in A$ together with a modified copy of $\mathcal{A}$. This modified copy of $\mathcal{A}$ has states of the following two forms:

- pairs (q, a) where q is a state of $\mathcal{A}$ and $a \in A$;
- triples (d, q, a) where $d \in \{\mathtt{D}_1, \ldots, \mathtt{D}_\ell\}$, q is a state of $\mathcal{A}$, and $a \in A$.

Given a transition $\delta(q, a)$ of the automaton $\mathcal{A}$ and a vector $\vec{a} \in A^\ell$ let $\bar{\delta}(q, a, \vec{a})$ be defined as the same formula as $\delta(q, a)$, except that each atom (d, q) is replaced by $\big(\mathtt{R}, (d, q, \vec{a}(d))\big)$ — a transition to the right in a tree to the state $(d, q, \vec{a}(d))$ of $\mathcal{A}_\mathcal{P}$. Now, the automaton $\mathcal{A}_\mathcal{P}$, together with all the transitions of $\mathcal{A}_{\tau(a)}$ for $a \in A$ has the following transitions for $b \in A_0$:

$$\delta\big((q,a), b\big) \stackrel{\text{def}}{=} \bigvee_{\vec{a} \in A^\ell} \big(\mathtt{L}, j(\vec{a})\big) \wedge \bar{\delta}(q, a, \vec{a})$$

where $j(\vec{a})$ is the respective state of the automaton $\mathcal{A}_{\tau(a)}$ such that $\mu_0\big(\mathrm{L}(\mathcal{A}_{\tau(a);}, j(\vec{a})\big) = \tau(a)(\vec{a})$

$$\delta\big((\mathtt{D}_1, q, a), b\big) \stackrel{\text{def}}{=} \big(\mathtt{L}, (q, a)\big)$$

$$\delta\big((\mathtt{D}_{k+1}, q, a), b\big) \stackrel{\text{def}}{=} \big(\mathtt{R}, (\mathtt{D}_k, q, a)\big) \qquad \text{for } k = 1, \ldots, \ell{-}1.$$

The priority mapping of $\mathcal{A}_\mathcal{P}$ is taken from $\mathcal{A}_{\tau(a)}$ and $\mathcal{A}$ respectively, i.e. $\Omega(q, a) = \Omega(d, q, a) = \Omega(q) + 2$ — we need this shift because the initial states of $\mathcal{A}_{\tau(a)}$ have priority 2. Let the initial state of $\mathcal{A}_\mathcal{P}$ be $(q_{\mathrm{I}}, a_{\mathrm{I}})$.

The automaton $\mathcal{A}_\mathcal{P}$ is designed in such a way to treat each tree $t \in \mathrm{Tr}_{A_0}$ as an encoded version of a tree $t \in \mathrm{Tr}_A$. To formally prove this fact, we first need to define that encoding. For this purpose, we define a family of functions T_a from Tr_{A_0} into $\mathrm{Tr}_A^{(\ell)}$ indexed by letters $a \in A$. Consider $a \in A$ and a tree $t \in \mathrm{Tr}_{A_0}$. Let $t' \stackrel{\text{def}}{=} t{\restriction}_{\mathtt{L}}$ be the left subtree of t. Similarly, for $k = 1, \ldots, \ell$ let $t_k \stackrel{\text{def}}{=} t{\restriction}_{\mathtt{R}^k\mathtt{L}}$. Let $\vec{a} \in A^\ell$ be the unique vector of letters such that $t' \in \mathrm{L}\big(\mathcal{A}_{\tau(a)}, j(\vec{a})\big)$. Notice that since t was chosen randomly, the probability distribution of the vectors $\vec{a}$ defined here is exactly $\tau(a)$. Then, let the resulting tree $T_a(t)$ have the root labelled a and for $k = 1, \ldots, \ell$ let the $\mathtt{D}_k$-th subtree of $T_a(t)$ equal $T_{\vec{a}(k)}(t_k)$. See Figure 3 for a depiction of that definition.

▷ Claim 27. Given a tree $t \in \mathrm{Tr}_{A_0}$ the automaton $\mathcal{A}_\mathcal{P}$ accepts t from a state (q, a) if and only if $\mathcal{A}$ accepts the tree $T_a(t)$ from q. In other words,

$$\mathrm{L}\big(\mathcal{A}_\mathcal{P}, (q,a)\big) = T_a^{-1}\big(\mathrm{L}(\mathcal{A}, q)\big).$$

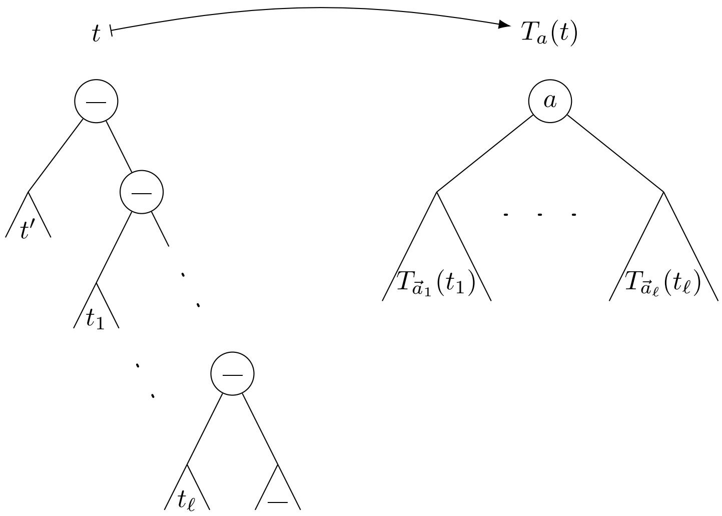

Figure 3 An illustration of an operation T_a for $a \in A$. Nodes and the subtree marked with _ are irrelevant in this construction. The subtree t' is used to determine which vector $\vec{a} \in A^\ell$ to use — it simulates the random choice of that vector using $\tau(a)$. Then the subtrees t_k for $k = 1, \ldots, \ell$ are recursively decoded by $T_{\vec{a}_k}$ according to the chosen letters of $\vec{a}$.

Proof. First observe that Lemma 26 implies that whenever $\mathcal{A}_\mathcal{P}$ takes a transition of the form $\delta\big((q,a),b\big)$ then there is exactly one candidate of $\vec{a} \in A^\ell$ such that the left subtree under the current node can be accepted from the state $j(\vec{a})$. Therefore, player $\exists$ in the game $\mathcal{G}\big(t,(q,a)\big)$ is always forced to choose that disjunct there. If the proper disjunct is chosen, then the choice of the atom $\big(\mathrm{L}, j(\vec{a})\big)$ is losing for $\forall$ because the respective subtree t' belongs to $\mathrm{L}\big(\mathcal{A}_{\tau(a)}, j(\vec{a})\big)$. Thus, we can assume that $\forall$ never chooses this atom.

Under the two above assumptions, the game $\mathcal{G}\big(t,(q,a)\big)$ given by the automaton $\mathcal{A}_\mathcal{P}$ becomes equivalent to the game $\mathcal{G}(T_a(t), q)$ given by the automaton $\mathcal{A}$. ◀

The next lemma states that the mapping T_{a_I} for the initial symbol $a_\mathrm{I} \in A$ allows to move between the measures μ_0 and $\mu_\mathcal{P}$. Recall that we have assumed that $\alpha_\mathrm{I}(a_\mathrm{I}) = 1$.

▶ **Lemma 28.** *The mapping T_{a_I} preserves the measure: for every measurable subset $L \subseteq \mathrm{Tr}_A^{(\ell)}$ and its pre-image $L' \stackrel{def}{=} T_{a_\mathrm{I}}^{-1}(L)$ we have $\mu_0(L') = \mu_\mathcal{P}(L)$.*

Proof. It is enough to check this on a basic set L as in (21). But in that case it follows from Lemma 26 and the fact that the subtrees t' used to choose the respective vectors $\vec{a}$ have pairwise-incomparable roots. ◀

By applying Claim 27 and Lemma 28 we obtain that

$$\mu_0\big(\mathrm{L}(\mathcal{A}_\mathcal{P})\big) = \mu_0\big(\mathrm{L}(\mathcal{A}_\mathcal{P}, (q_\mathrm{I}, a_\mathrm{I}))\big) = \mu_0\big(T_{a_\mathrm{I}}^{-1}(\mathrm{L}(\mathcal{A}, q_\mathrm{I}))\big) = \mu_\mathcal{P}\big(\mathrm{L}(\mathcal{A}, q_\mathrm{I})\big) = \mu_\mathcal{P}\big(\mathrm{L}(\mathcal{A})\big).$$

This concludes the proof of Proposition 25.

8.2 Branching processes - direct construction

In this section we want to show how to extend our main result, of computing the uniform measure of a weak-MSO recognisable language, to measures generated by arbitrary branching processes.

The core of the proof will stay the same as in the main part of the article, we will define two types of operators $\mathcal{F}$, $\mathcal{Q}$, and explain, how the measure can be computed using those operators.

Let us fix a regular language of trees L and a weak alternating automaton $\mathcal{A}$ such that $\mathrm{L}(\mathcal{A}) = L$.

Let us fix a branching process $\mathcal{P} = \langle A, \tau, \alpha_{\mathrm{I}}\rangle$. We want to distinguish between the alphabet A treated as the set of labels of trees, from A treated as vertices of the branching process $\mathcal{P}$. Thus, we put $V = A$ and use the symbol $v \in V$ to denote letters generated by $\mathcal{P}$. This means that $\tau\colon V \to \mathcal{D}V^2$ and $\alpha_{\mathrm{I}} \in \mathcal{D}V$.

By $\mu_{\mathcal{P}}(v)$, where $v \in V$, we understand the measure induced by the process $\mathcal{P}$ with the initial distribution α'_{I} concentrated in v, i.e. $\alpha'_{\mathrm{I}}(v) = 1$ and $\alpha'_{\mathrm{I}}(v') = 0$ for $v' \neq v$.

By a simple calculation, we have that

$$\mu_{\mathcal{P}}(L) = \sum_{v\in V} \alpha_{\mathrm{I}}(v) \cdot \mu_{\mathcal{P}}(v)(L). \tag{22}$$

Thus, we only need to determine the values of $\vec{\mu}_{\mathcal{P}}(v)(L)$ for $v \in V$.

The measure defined in a subtree, unlike in Remark 3, is not always uniform and may non-trivially depend on the label of the root of the subtree. This implies that the distributions β used in the whole procedure may depend on the initial vertex and, thus, this information has to be included. It turns out that simply lifting distributions to tuples indexed by the origin point in the branching process is enough. We lift distributions to tuples of distributions by defining $\beta_{\mathcal{P}}\colon V \to \mathcal{D}\mathsf{P}(Q)$. In other words, the basic space that we work is, instead of $\mathcal{D}\mathsf{P}(Q)$ is now $\big(V \to \mathcal{D}\mathsf{P}(Q)\big)$. Let the order $\preceq$ be defined on $V \to \mathcal{D}\mathsf{P}(Q)$ coordinate-wise: $\alpha_{\mathcal{P}} \preceq \beta_{\mathcal{P}}$ if for every $v \in V$ we have $\alpha_{\mathcal{P}}(v) \preceq \beta_{\mathcal{P}}(v)$.

Now, our definitions of previously used operations have to be adjusted accordingly. By slight abuse of notation, we will simply overload the definitions. This will not produce confusion, since we will not use the old definitions in this part.

Take a Q-indexed family $\mathcal{L}$. Define the distribution $\vec{\mu}_{\mathcal{P}} \in V \to \mathcal{D}P(Q)$.

$$\vec{\mu}_{\mathcal{P}}\big(\mathcal{L}\big)(v)(P) \overset{\mathrm{def}}{=} \mu_{\mathcal{P}}(v)\big\{t \in \mathrm{Tr}_A \mid \mathcal{L}[t] = P\big\} \tag{23}$$

Notice that the set of trees with root labelled v is of full $\mu_{\mathcal{P}}(v)$ measure. Thus

$$\mu_{\mathcal{P}}(v)\big\{t \in \mathrm{Tr}_A \mid \mathcal{L}[t] = P\big\} = \mu_{\mathcal{P}}(v)\big\{t \in \mathrm{Tr}_A \mid \mathcal{L}[t] = P \wedge t(\varepsilon) = v\big\} \tag{24}$$

Also, the measure $\mu_{\mathcal{P}}$ satisfies the following independence property similar to Remark 3.

▶ Remark 29. Let $L_{\mathtt{L}}, L_{\mathtt{R}} \subseteq \mathrm{Tr}_A$ be two Borel sets and $v \in V$. Then

$$\mu_{\mathcal{P}}(v)\big\{t \mid t{\restriction}_{\mathtt{L}}{\in}L_{\mathtt{L}} \wedge t(\varepsilon){=}v \wedge t{\restriction}_{\mathtt{R}}{\in}L_{\mathtt{R}}\big\} = \sum_{v_{\mathtt{L}},v_{\mathtt{R}}\in V^2} \tau(v)(v_{\mathtt{L}}, v_{\mathtt{R}}) \cdot \mu_{\mathcal{P}}(v_{\mathtt{L}})\big(L_{\mathtt{L}}\big) \cdot \mu_{\mathcal{P}}(v_{\mathtt{R}})\big(L_{\mathtt{R}}\big).$$

As before, the sets in consideration are measurable thanks to Proposition 6.

▶ **Lemma 30.** *Fix $v \in V$. If for every $q \in Q$ we have $\mathcal{L}(q) \subseteq \mathcal{L}'(q)$ then $\vec{\mu}_{\mathcal{P}}(\mathcal{L}) \preceq \vec{\mu}_{\mathcal{P}}(\mathcal{L}')$ in $V \to \mathcal{D}\mathsf{P}(Q)$.*

The proof is the same as the proof of Lemma 14, as it depends on general properties of measures.

Now, we examine the sequences of distributions $\vec{\mu}_{\mathcal{P}}\big(\mathcal{S}^n_i\big)$, $\vec{\mu}_{\mathcal{P}}\big(\mathcal{R}^n_i\big)$, $\vec{\mu}_{\mathcal{P}}\big(\mathcal{S}^n_\infty\big)$, and $\vec{\mu}_{\mathcal{P}}\big(\mathcal{R}^n_\infty\big)$ arising from the Q-families introduced before. Our aim again is to bind them by some equations computable within $V \to \mathcal{D}\mathsf{P}(Q)$. As an analogue to the operation $\mathcal{F}$, we introduce the function $\mathcal{F}_{\mathcal{P}}\colon (V \to \mathcal{D}\mathsf{P}(Q)) \to (V \to \mathcal{D}\mathsf{P}(Q))$ defined for $v \in V$, $\beta_{\mathcal{P}} \in V \to \mathcal{D}\mathsf{P}(Q)$, and $P \in \mathsf{P}(Q)$ by

$$\mathcal{F}_{\mathcal{P}}(\beta_{\mathcal{P}})(v)(P) \stackrel{\text{def}}{=} \sum_{(P_{\mathtt{L}},v,P_{\mathtt{R}})\in\Delta^{-1}(P)} \; \sum_{(v_{\mathtt{L}},v_{\mathtt{R}})\in V^2} \tau(v)(v_{\mathtt{L}},v_{\mathtt{R}})\big(\beta_{\mathcal{P}}(v_{\mathtt{L}})(P_{\mathtt{L}}) \cdot \beta_{\mathcal{P}}(v_{\mathtt{R}})(P_{\mathtt{R}})\big) \tag{25}$$

As before, the formula guarantees that $\mathcal{F}_{\mathcal{P}}(\beta_{\mathcal{P}})(v)$ is indeed a probabilistic distribution in $\mathcal{D}\mathsf{P}(Q)$. The operator $\mathcal{F}$ will allow us to transfer the inductive definitions of the Q-families $\mathcal{S}^n_{i+1}$ and $\mathcal{R}^n_{i+1}$ given by Lemma 12, to the level of probability distributions.

From now on, we omit the index $\mathcal{P}$ in $\mathcal{F}_{\mathcal{P}}$.

▶ **Lemma 31.** *For each $n \in \mathbb{N}$ and $i \in \mathbb{N}$ we have*

$$\vec{\mu}_{\mathcal{P}}\big(\mathcal{S}^n_{i+1}\big) = \mathcal{F}\Big(\mu_{\mathcal{P}}\big(\mathcal{S}^n_i\big)\Big) \text{ and } \vec{\mu}_{\mathcal{P}}\big(\mathcal{R}^n_{i+1}\big) = \mathcal{F}\Big(\vec{\mu}_{\mathcal{P}}\big(\mathcal{R}^n_i\big)\Big).$$

Proof. Take $P \in \mathsf{P}(Q)$ and $v \in V$ observe that

$$\begin{aligned}
\mathcal{F}\Big(\vec{\mu}_{\mathcal{P}}\big(\mathcal{S}^n_i\big)\Big)(v)(P) &\stackrel{(1)}{=} \sum_{(P_{\mathtt{L}},v,P_{\mathtt{R}})\in\Delta^{-1}(P)} \; \sum_{(v_{\mathtt{L}},v_{\mathtt{R}})\in V^2} \tau(v)(v_{\mathtt{L}},v_{\mathtt{R}})\cdot \\
&\qquad \Big(\vec{\mu}_{\mathcal{P}}\big(\mathcal{S}^n_i\big)(v_{\mathtt{L}})(P_{\mathtt{L}}) \cdot \vec{\mu}_{\mathcal{P}}\big(\mathcal{S}^n_i\big)(v_{\mathtt{R}})(P_{\mathtt{R}})\Big) \\
&\stackrel{(2)}{=} \sum_{(P_{\mathtt{L}},v,P_{\mathtt{R}})\in\Delta^{-1}(P)} \; \sum_{(v_{\mathtt{L}},v_{\mathtt{R}})\in V^2} \tau(v)(v_{\mathtt{L}},v_{\mathtt{R}})\cdot \\
&\qquad \Big(\mu_{\mathcal{P}}(v_{\mathtt{L}})\big\{t_{\mathtt{L}} \mid \mathcal{S}^n_i[t_{\mathtt{L}}]{=}P_{\mathtt{L}}\big\} \cdot \mu_{\mathcal{P}}(v_{\mathtt{R}})\big\{t_{\mathtt{R}} \mid \mathcal{S}^n_i[t_{\mathtt{R}}]{=}P_{\mathtt{R}}\big\}\Big) \\
&\stackrel{(3)}{=} \sum_{(P_{\mathtt{L}},v,P_{\mathtt{R}})\in\Delta^{-1}(P)} \mu_{\mathcal{P}}(v)\big\{t \mid \mathcal{S}^n_i[t{\restriction}_{\mathtt{L}}]{=}P_{\mathtt{L}} \wedge t(\varepsilon){=}v \wedge \mathcal{S}^n_i[t{\restriction}_{\mathtt{R}}]{=}P_{\mathtt{R}}\big\} \\
&\stackrel{(4)}{=} \mu_{\mathcal{P}}(v)\left(\bigcup_{(P_{\mathtt{L}},v,P_{\mathtt{R}})\in\Delta^{-1}(P)} \big\{t \mid \mathcal{S}^n_i[t{\restriction}_{\mathtt{L}}]{=}P_{\mathtt{L}} \wedge t(\varepsilon){=}v \wedge \mathcal{S}^n_i[t{\restriction}_{\mathtt{R}}]{=}P_{\mathtt{R}}\big\}\right) \\
&\stackrel{(5)}{=} \mu_{\mathcal{P}}(v)\Big\{t \in \mathrm{Tr}_A \mid \Delta\big(\mathcal{S}^n_i[t{\restriction}_{\mathtt{L}}], t(\varepsilon), \mathcal{S}^n_i[t{\restriction}_{\mathtt{R}}]\big){=}P \wedge t(\varepsilon) = v\Big\} \\
&\stackrel{(6)}{=} \mu_{\mathcal{P}}(v)\Big\{t \in \mathrm{Tr}_A \mid \mathcal{S}^n_{i+1}[t]{=}P \wedge t(\varepsilon) = v\Big\} \\
&\stackrel{(7)}{=} \mu_{\mathcal{P}}(v)\Big\{t \in \mathrm{Tr}_A \mid \mathcal{S}^n_{i+1}[t]{=}P\Big\} \stackrel{(8)}{=} \vec{\mu}_{\mathcal{P}}\big(\mathcal{S}^n_{i+1}\big)(v)(P),
\end{aligned}$$

where: (1) is just the definition of $\mathcal{F}\Big(\vec{\mu}_{\mathcal{P}}\big(\mathcal{S}^n_i\big)\Big)$; (2) follows from the definition of $\vec{\mu}_{\mathcal{P}}\big(\mathcal{S}^n_i\big)$; (3) follows from the definition of $\mu_{\mathcal{P}}$ and Remark 29; (4) follows from the fact that the measured sets are pairwise disjoint; (5) follows simply from the definition of Δ; (6) follows from Lemma 12; (7) follows from (24); and (7) is just the definition of $\vec{\mu}_{\mathcal{P}}\big(\mathcal{S}^n_{i+1}\big)$.

The proof for $\mathcal{R}^n_{i+1}$ is entirely analogous (we use the $\mathcal{R}^n_i$ variant of Lemma 12 instead). ◀

Now, recall that $Q_{\geq n}$ and $Q_{<n}$ are sets of states of respective priorities. Let the functions $\mathcal{Q}_{\geq n}, \mathcal{Q}_{<n}\colon (V \to \mathcal{D}\mathsf{P}(Q)) \to (V \to \mathcal{D}\mathsf{P}(Q))$ be defined by

$$\mathcal{Q}_{\geq n}(\beta_{\mathcal{P}})(v)(P) \stackrel{\text{def}}{=} \sum_{P'\colon\, P' \cup Q_{\geq n} = P} \beta_{\mathcal{P}}(v)(P'),$$

$$\mathcal{Q}_{<n}(\beta_{\mathcal{P}})(v)(P) \stackrel{\text{def}}{=} \sum_{P'\colon\, P' \cap Q_{<n} = P} \beta_{\mathcal{P}}(v)(P').$$

Again, the formulas guarantee that $\mathcal{Q}_{\geq n}(\beta_{\mathcal{P}})(v)$ and $\mathcal{Q}_{<n}(\beta_{\mathcal{P}})(v)$ are both probabilistic distributions in $\mathcal{D}\mathsf{P}(Q)$. The following lemma shows the relation between these functions and the limit distributions $\vec{\mu}_{\mathcal{P}}\big(\mathcal{S}^{n-1}_\infty\big)$ and $\vec{\mu}_{\mathcal{P}}\big(\mathcal{R}^{n-1}_\infty\big)$.

▶ **Lemma 32.** *For each $n \in \mathbb{N}$ we have*

$$\mathcal{Q}_{<n}\Big(\vec{\mu}_{\mathcal{P}}\big(\mathcal{S}^{n-1}_{\infty}\big)\Big) = \vec{\mu}_{\mathcal{P}}\big(\mathcal{R}^{n}_{0}\big) \qquad \text{if } n \text{ is odd,}$$

$$\mathcal{Q}_{\geq n}\Big(\vec{\mu}_{\mathcal{P}}\big(\mathcal{R}^{n-1}_{\infty}\big)\Big) = \vec{\mu}_{\mathcal{P}}\big(\mathcal{S}^{n}_{0}\big) \qquad \text{if } n \text{ is even.}$$

Proof. Note that the proof of Lemma 16 does not depend on the underlying measure, and therefore it carries over. ◀

Again, the two above lemmata show that the operators $\mathcal{F}$, $\mathcal{Q}_{<n}$, and $\mathcal{Q}_{\geq n}$ are enough to perform the respective computations in $V \to \mathcal{D}\mathsf{P}(Q)$ as they do on Figure 1.

In Section 6 we prove the connection between the limit distributions $\vec{\mu}_{\mathcal{P}}\big(\mathcal{S}^{n}_{\infty}\big)$, $\vec{\mu}_{\mathcal{P}}\big(\mathcal{R}^{n}_{\infty}\big)$ (unary versions) and fixed points of the operator $\mathcal{F}$, see Lemma 17. The same line of proof works in the case of tuples, if we apply the reasoning point-wise, i.e. we work now in $(\mathbb{R}^{\mathsf{P}(Q)})^{V} = \mathbb{R}^{\mathsf{P}(Q)\times V}$. The only missing ingredient is the monotonicity of the new operator $\mathcal{F}_{\mathcal{P}}$.

▶ **Lemma 33.** *The operator $\mathcal{F}\colon (V \to \mathcal{D}\mathsf{P}(Q)) \to (V \to \mathcal{D}\mathsf{P}(Q))$ is point-wise monotone in the order $\preceq$ and continuous in $\mathbb{R}^{\mathsf{P}(Q)\times V}$.*

Proof. Continuity is again trivial. The fact that $\mathcal{F}$ is monotone follows from the monotonicity of Δ and the point-wise definition of the order as follows. Recall the definition of $\mathcal{F}_{\mathcal{P}}$, cf. (25):

$$\mathcal{F}(\beta_{\mathcal{P}})(v)(P) = \sum_{(P_{\mathtt{L}},v,P_{\mathtt{R}})\in\Delta^{-1}(P)} \; \sum_{(v_{\mathtt{L}},v_{\mathtt{R}})\in V^2} \tau(v)(v_{\mathtt{L}},v_{\mathtt{R}})\big(\beta_{\mathcal{P}}(v_{\mathtt{L}})(P_{\mathtt{L}})\cdot\beta_{\mathcal{P}}(v_{\mathtt{R}})(P_{\mathtt{R}})\big)$$

We need to prove that for a fixed $v \in V$ function $\mathcal{F}$ is monotone w.r.t. the order $\preceq$. Thus, for every $\alpha_{\mathcal{P}} \preceq \beta_{\mathcal{P}} \in V \to \mathcal{D}\mathsf{P}(Q)$ and an upward-closed family $U \subseteq \mathsf{P}(Q)$ we should have $\sum_{P\in U}\mathcal{F}_{\mathcal{P}}(\alpha_{\mathcal{P}})(v)(P) \leq \sum_{P\in U}\mathcal{F}_{\mathcal{P}}(\beta_{\mathcal{P}})(v)(P)$. After splitting the sum over separate letters v, $v_{\mathtt{L}}$, $v_{\mathtt{R}} \in V$, it is enough to show that for $O_v \stackrel{\text{def}}{=} \{(P_{\mathtt{L}},P_{\mathtt{R}}) \mid \Delta(P_{\mathtt{L}},v,P_{\mathtt{R}}) \in U\}$ we have

$$\sum_{(P_{\mathtt{L}},P_{\mathtt{R}})\in O_v} \alpha_{\mathcal{P}}(v_{\mathtt{L}})(P_{\mathtt{L}})\cdot\alpha_{\mathcal{P}}(v_{\mathtt{R}})(P_{\mathtt{R}}) \leq \sum_{(P_{\mathtt{L}},P_{\mathtt{R}})\in O_v} \beta_{\mathcal{P}}(v_{\mathtt{L}})(P_{\mathtt{L}})\cdot\beta_{\mathcal{P}}(v_{\mathtt{R}})(P_{\mathtt{R}}).$$

The set O_v is again upward-closed on both coordinates, as in the proof of Lemma 18. We use the notation used there to denote the sections of that set. Thus, using the assumption that $\alpha_{\mathcal{P}} \preceq \beta_{\mathcal{P}}$ twice (once for $v_{\mathtt{L}}$ and once for $v_{\mathtt{R}}$), we obtain

$$\begin{aligned}
\sum_{(P_{\mathtt{L}},P_{\mathtt{R}})\in O_v} \alpha_{\mathcal{P}}(v_{\mathtt{L}})(P_{\mathtt{L}})\cdot\alpha_{\mathcal{P}}(v_{\mathtt{R}})(P_{\mathtt{R}}) &= \sum_{P_{\mathtt{L}}\in\mathsf{P}(Q)} \alpha_{\mathcal{P}}(v_{\mathtt{L}})(P_{\mathtt{L}})\cdot\left(\sum_{P_{\mathtt{R}}\in P_{\mathtt{L}}^{-1}\cdot O_v} \alpha_{\mathcal{P}}(v_{\mathtt{R}})(P_{\mathtt{R}})\right)\\
&\leq \sum_{P_{\mathtt{L}}\in\mathsf{P}(Q)} \alpha_{\mathcal{P}}(v_{\mathtt{L}})(P_{\mathtt{L}})\cdot\left(\sum_{P_{\mathtt{R}}\in P_{\mathtt{L}}^{-1}\cdot O_v} \beta_{\mathcal{P}}(v_{\mathtt{R}})(P_{\mathtt{R}})\right)\\
&= \sum_{(P_{\mathtt{L}},P_{\mathtt{R}})\in O_v} \alpha_{\mathcal{P}}(v_{\mathtt{L}})(P_{\mathtt{L}})\cdot\beta_{\mathcal{P}}(v_{\mathtt{R}})(P_{\mathtt{R}})\\
&= \sum_{(P_{\mathtt{L}},P_{\mathtt{R}})\in O_v} \beta_{\mathcal{P}}(v_{\mathtt{R}})(P_{\mathtt{R}})\cdot\alpha_{\mathcal{P}}(v_{\mathtt{L}})(P_{\mathtt{L}})\\
&= \sum_{P_{\mathtt{R}}\in\mathsf{P}(Q)} \beta_{\mathcal{P}}(v_{\mathtt{R}})(P_{\mathtt{R}})\cdot\left(\sum_{P_{\mathtt{L}}\in O_v\cdot P_{\mathtt{R}}^{-1}} \alpha_{\mathcal{P}}(v_{\mathtt{L}})(P_{\mathtt{L}})\right)
\end{aligned}$$

$$\leq \sum_{P_{\mathrm{R}} \in \mathrm{P}(Q)} \beta_{\mathcal{P}}(v_{\mathrm{R}})(P_{\mathrm{R}}) \cdot \left(\sum_{P_{\mathrm{L}} \in O_v \cdot P_{\mathrm{R}}^{-1}} \beta_{\mathcal{P}}(v_{\mathrm{L}})(P_{\mathrm{L}}) \right)$$
$$= \sum_{(P_{\mathrm{L}}, P_{\mathrm{R}}) \in O_v} \beta_{\mathcal{P}}(v_{\mathrm{R}})(P_{\mathrm{R}}) \cdot \beta_{\mathcal{P}}(v_{\mathrm{L}})(P_{\mathrm{L}})$$
$$= \sum_{(P_{\mathrm{L}}, P_{\mathrm{R}}) \in O_v} \beta_{\mathcal{P}}(v_{\mathrm{L}})(P_{\mathrm{L}}) \cdot \beta_{\mathcal{P}}(v_{\mathrm{R}})(P_{\mathrm{R}}).$$

◀

Since $\mathcal{F}$ is continuous and monotone, $\vec{\mu}_{\mathcal{P}}(\mathcal{S}^n_\infty)$ and $\vec{\mu}_{\mathcal{P}}(\mathcal{R}^n_\infty)$ are the greatest-and least-fixed points of the appropriate operations. This observation allows us to compute the values $\mu_{\mathcal{P}}(v)(L)$ and given Equation (22) we obtain the measure $\mu_{\mathcal{P}}(L)$.

9 Representing algebraic numbers

We now use the formulae $\psi_{\mathcal{A}}$ and $\psi_{\mathcal{A},\mathcal{P}}$ constructed above to find the measure of the language $\mathrm{L}(\mathcal{A})$. We use the celebrated result of Tarski [25] and its two algorithmic improvements.

▶ **Theorem 34** ([2, 3]). *Given a formula ψ of first-order logic over $\mathbb{R}$, one can decide if ψ holds in deterministic exponential space. Moreover, if ψ is in a prenex normal form and the alternation of quantifiers $\forall$ and $\exists$ in ψ is bounded then the algorithm works in single exponential time in the size of ψ.*

Proof of Theorem 2. Input a weak alternating automaton $\mathcal{A}$, a branching process $\mathcal{P}$, and a rational number q. Consider the formula $\psi \equiv \exists x.\ \psi_{\mathcal{A},\mathcal{P}}(x) \wedge q \bowtie x$, where $\bowtie$ is one of $<$, $=$, or $>$. Notice that ψ is in prenex normal form; its size is exponential in the size of $\mathcal{A}$ and polynomial in the size of $\mathcal{P}$; and its quantifier alternation is constant. Apply the algorithm from Theorem 34 to check whether ψ is true in $\mathbb{R}$. ◀

We can also compute a representation of the measure $\mu_{\mathcal{P}}(\mathrm{L}(\mathcal{A}))$. The *quantifier elimination* procedure due to Tarski [25] transforms a formula $\psi(x_1, \ldots, x_n)$ into an equivalent quantifier-free formula $\widehat{\psi}(x_1, \ldots, x_n)$, which moreover can be represented by a *semialgebraic* set, see [4, Chapter 2].

▶ **Theorem 35** ([7]). *Given a formula $\psi(x_1, \ldots, x_n)$ of first-order logic over $\mathbb{R}$, one can construct a representation of the set of tuples $(x_1, \ldots, x_n)$ satisfying ψ, as a semialgebraic set. Moreover, this algorithm works in time doubly-exponential in the size of ψ.*

Theorems 20 and 24 together with the above results imply the following claim.

▶ **Corollary 36.** *Given a weak alternating automaton $\mathcal{A}$ of size n, one can compute a representation of the value $\mu_0(\mathrm{L}(\mathcal{A}))$ as a singleton semialgebraic set in time triply exponential in n. Moreover, given a branching process of size m, one can compute a representation of the value $\mu_{\mathcal{P}}(\mathrm{L}(\mathcal{A}))$ as a singleton semialgebraic set in time triply exponential in n and doubly exponential in m.*

10 Conclusions

We have shown how to compute the probability measure of a tree language L recognised by a weak alternating automaton. The crucial trait is *continuity* of certain approximations of the measure of L in a properly chosen order $\preceq$, see Lemma 17. This continuity relies on König's lemma, cf. Lemma 9. In terms of μ-calculus, it stems from both the absence of alternation between least and greatest fixed points in formulae and the boundedness of branching in models (for a study of continuity in μ-calculus see [11]).

Whether our techniques can be extended beyond weak automata—hopefully to all tree automata or, equivalently, full MSO logic, or full μ-calculus—remains open. The question is of interest as, e.g. translation of the logic CTL* into μ-calculus requires at least one alternation between least and greatest fixed points (cf. [9], Exercise 10.13). On the other hand, fixed point formulas over binary trees are not continuous in general, and may require ω_1 iterations to reach stabilisation, already on the second level of the fixed-point hierarchy.

This problem has been already successfully tackled in the context of measurability of regular tree languages—Mio [16] uses Martin's axiom to control the behaviour of measure when taking limits of sequences of length ω_1. Such behaviour cannot be directly simulated in $\mathcal{D}X$, because each well-founded chain of distributions has a countable length. However, this need not be an absolute obstacle as it might be the case that the values of the measure of the iterations stabilise before the actual fixed point is reached, possibly in ω steps.

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