Unambiguous languages exhaust the index hierarchy

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POLAND

Non-determinism and unambiguity

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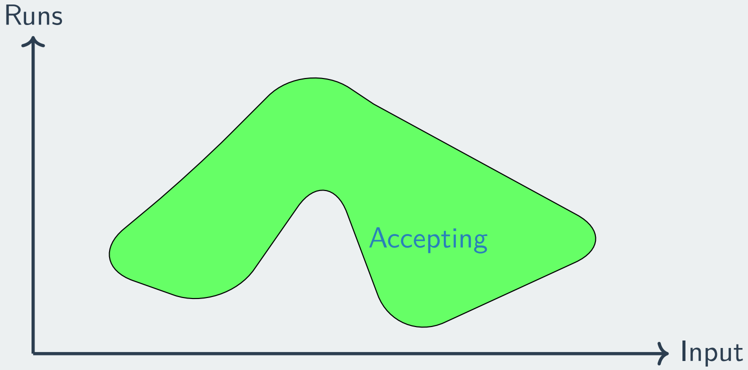

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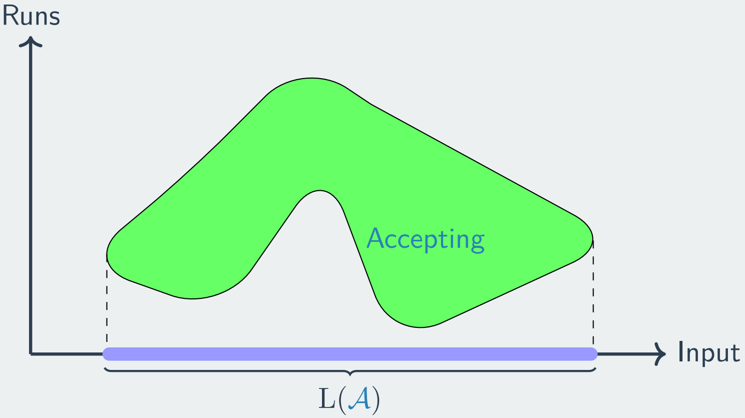

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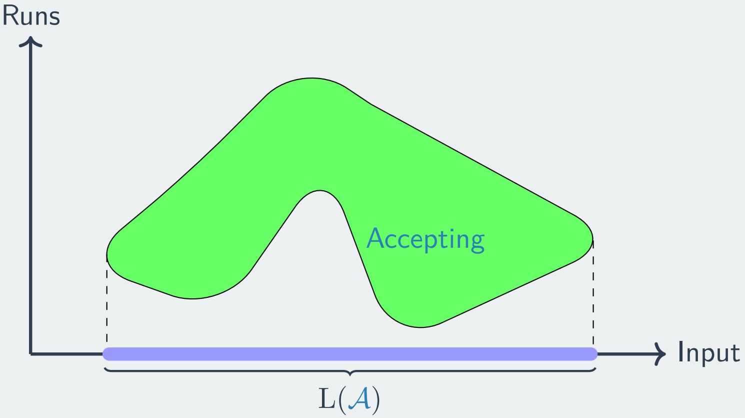

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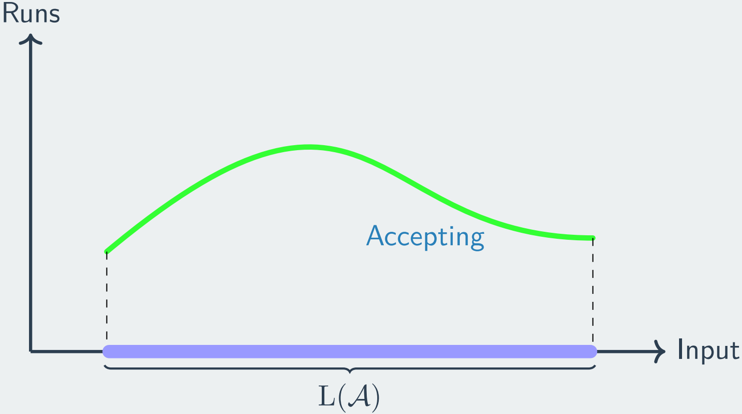

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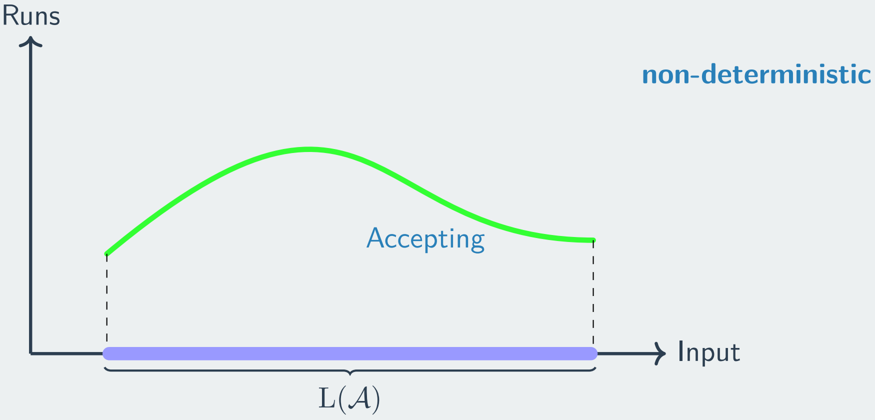

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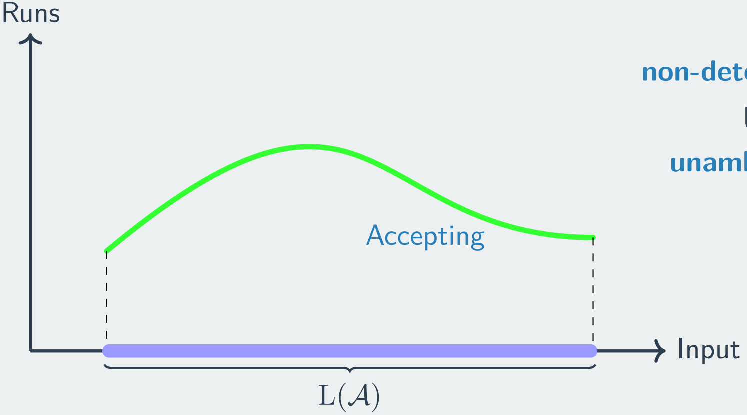

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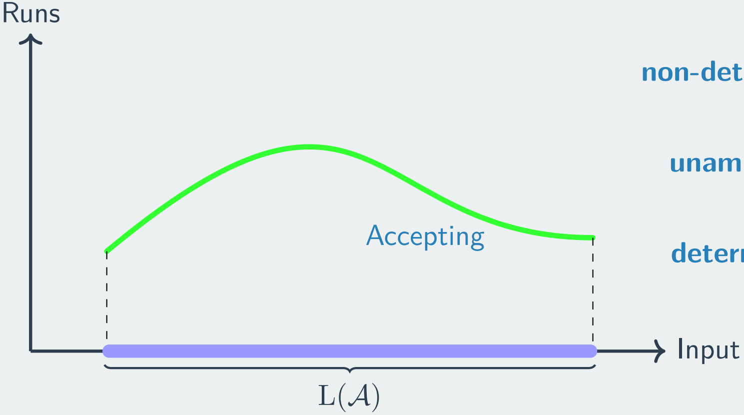

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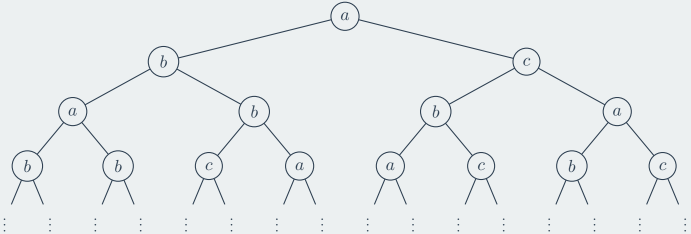

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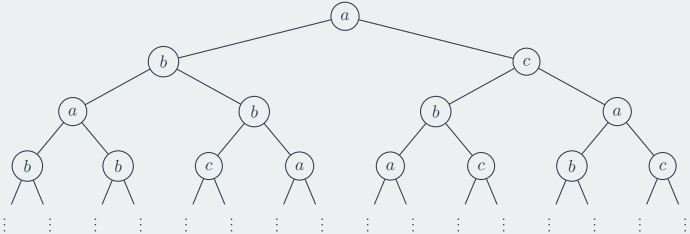

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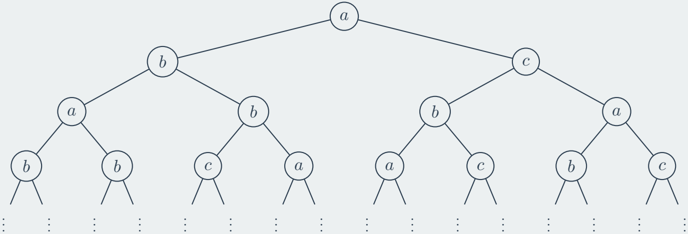

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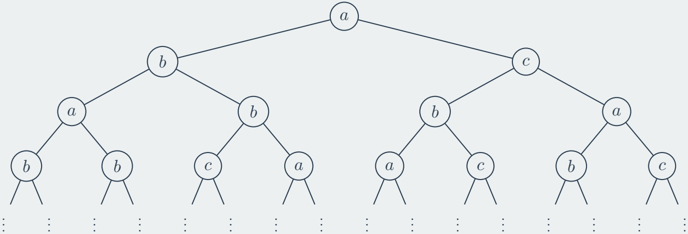

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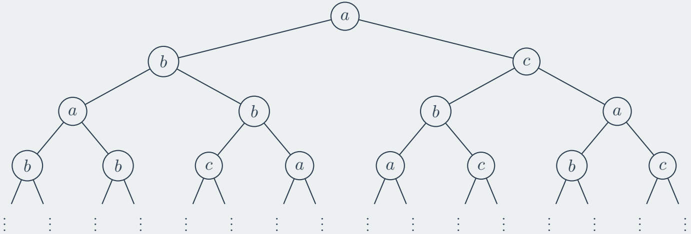

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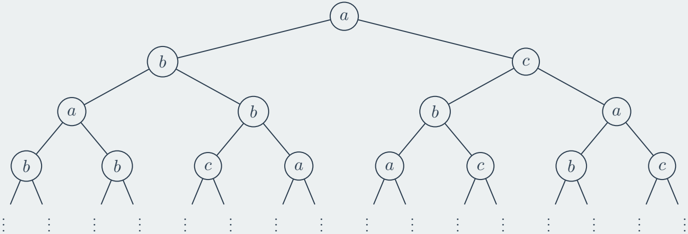

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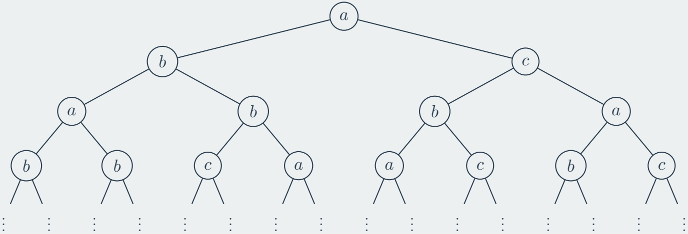

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Proof: Non-deterministic parity tree automata

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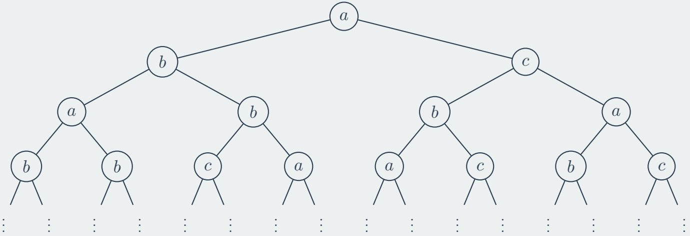

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$$\Uparrow$$

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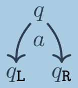
Non-deterministic parity tree automata

local consistency



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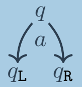

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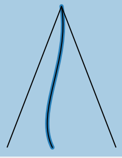

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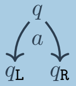

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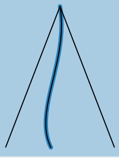

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  ⇝ Duparc + Fournier + Hummel + **this work**

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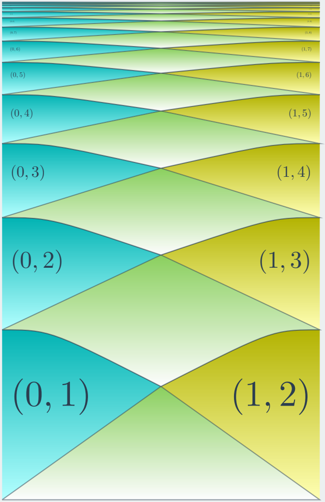

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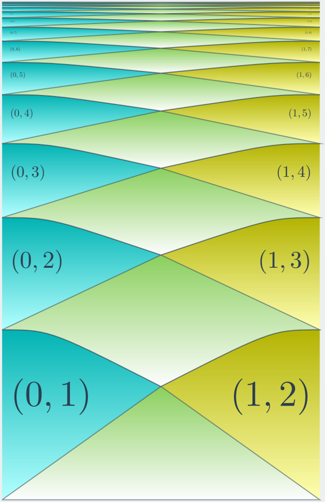

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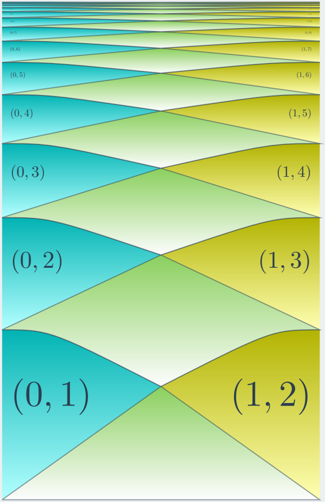

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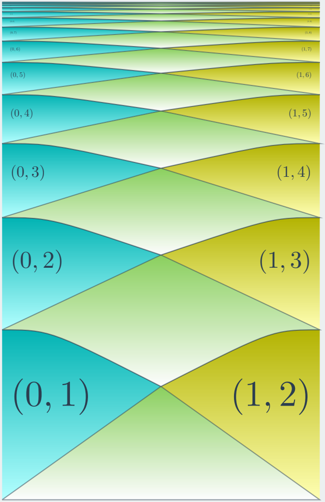

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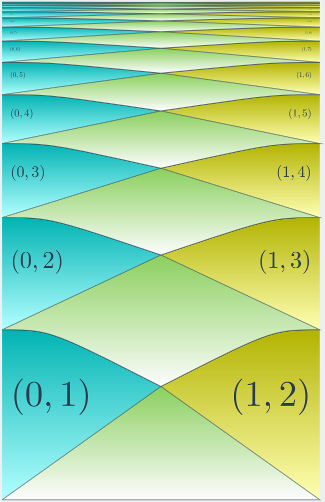
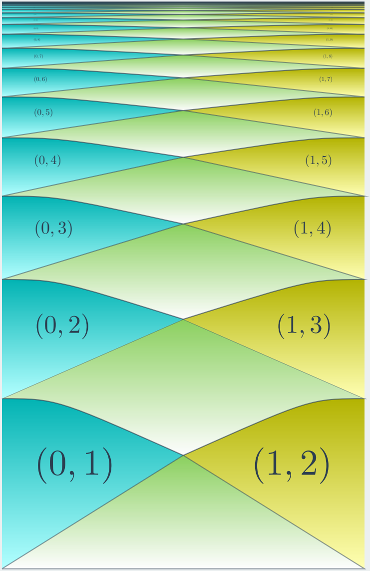


Alternating index hierarchy & unambiguous languages



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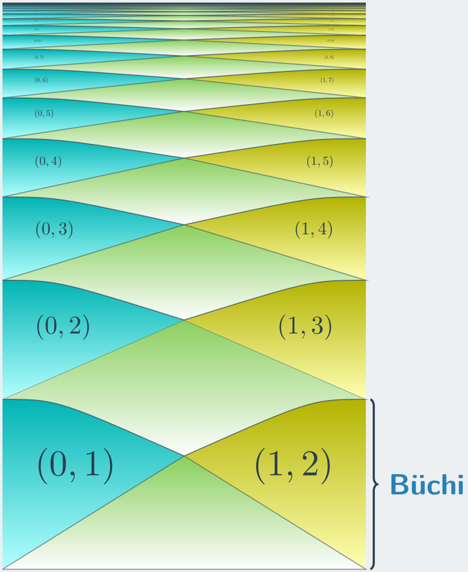

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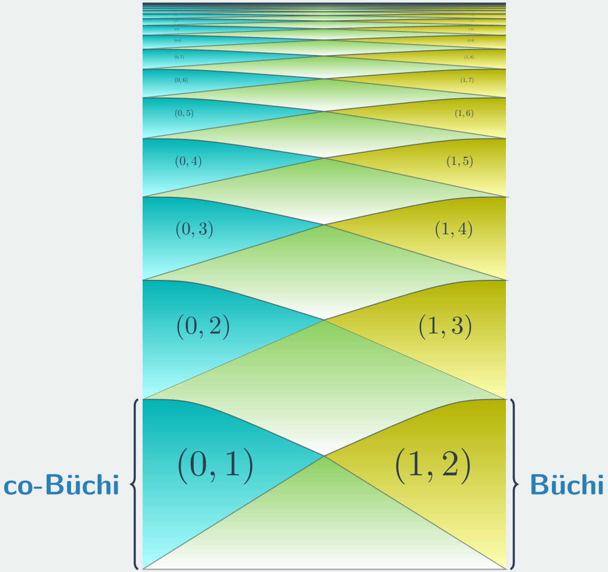

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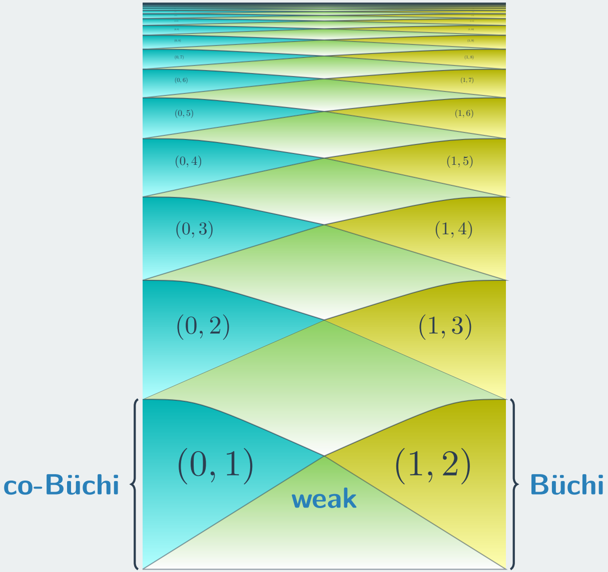

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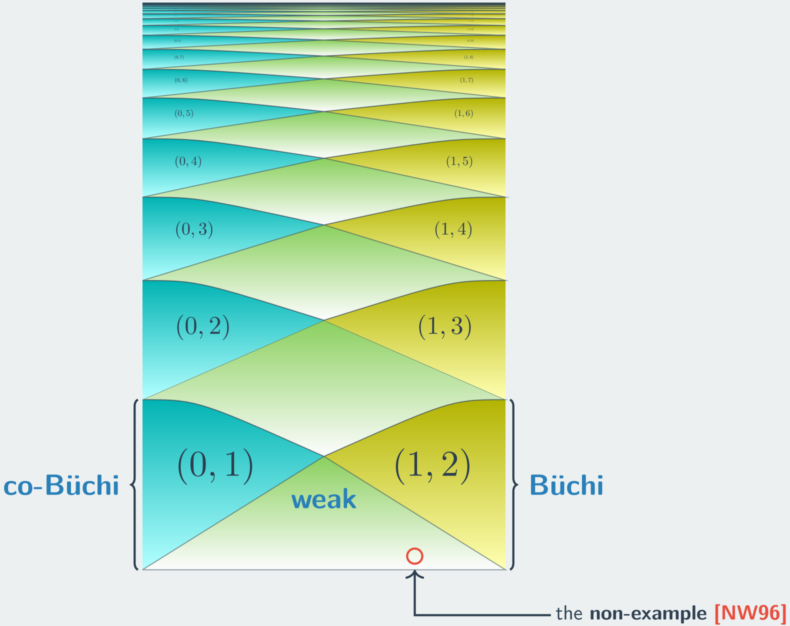

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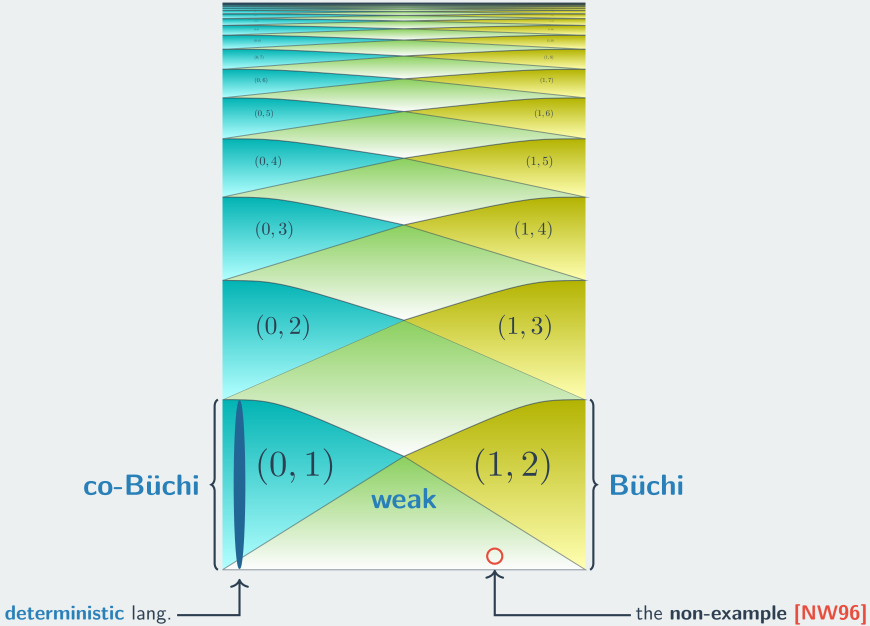

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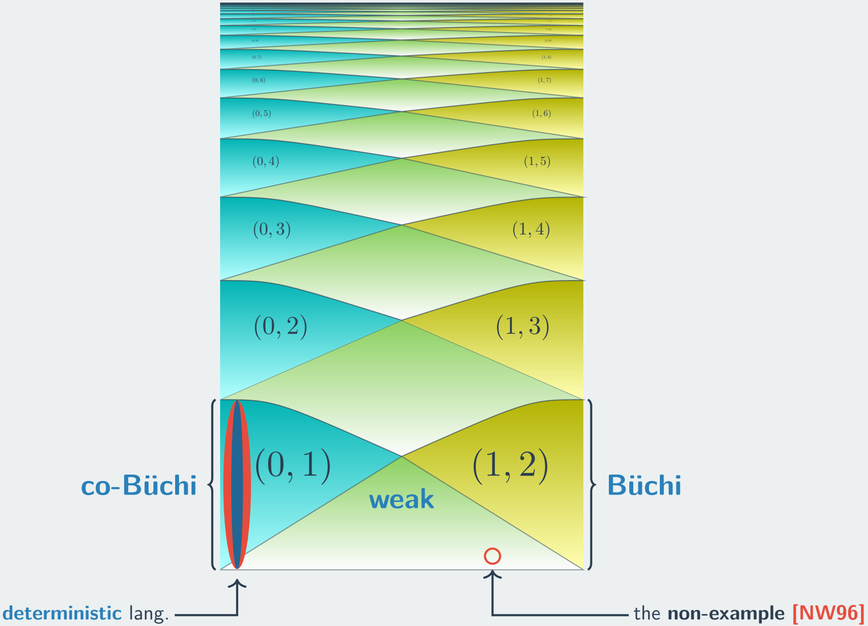

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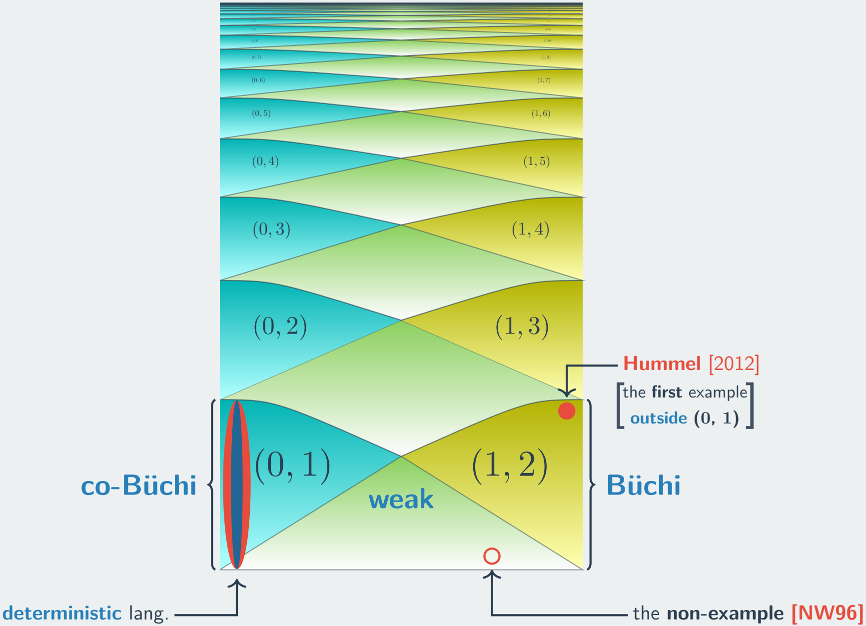

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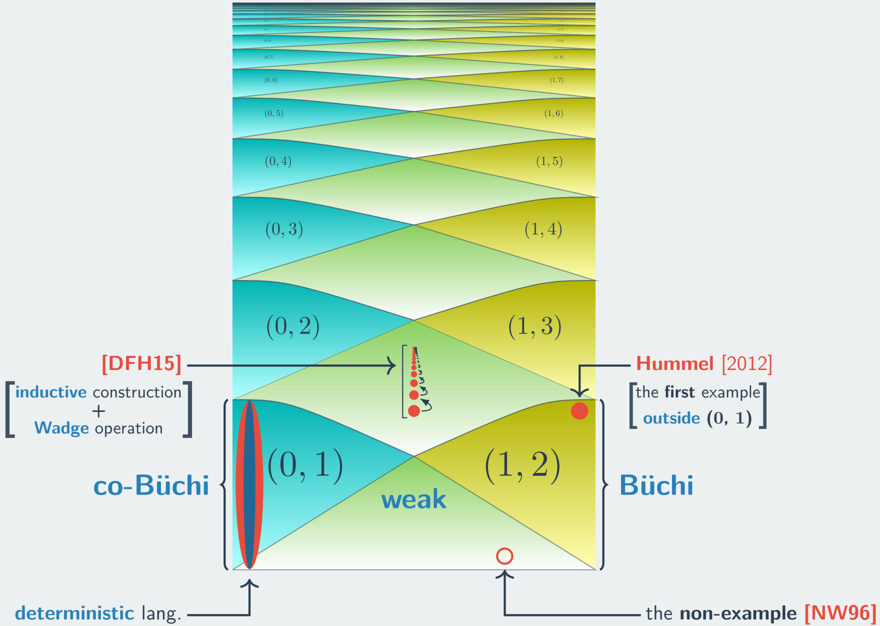

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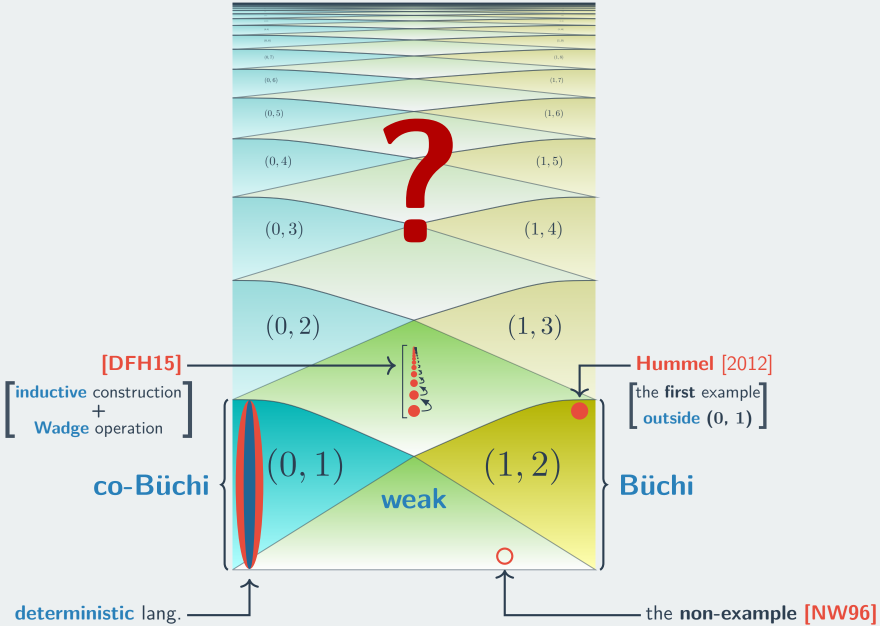

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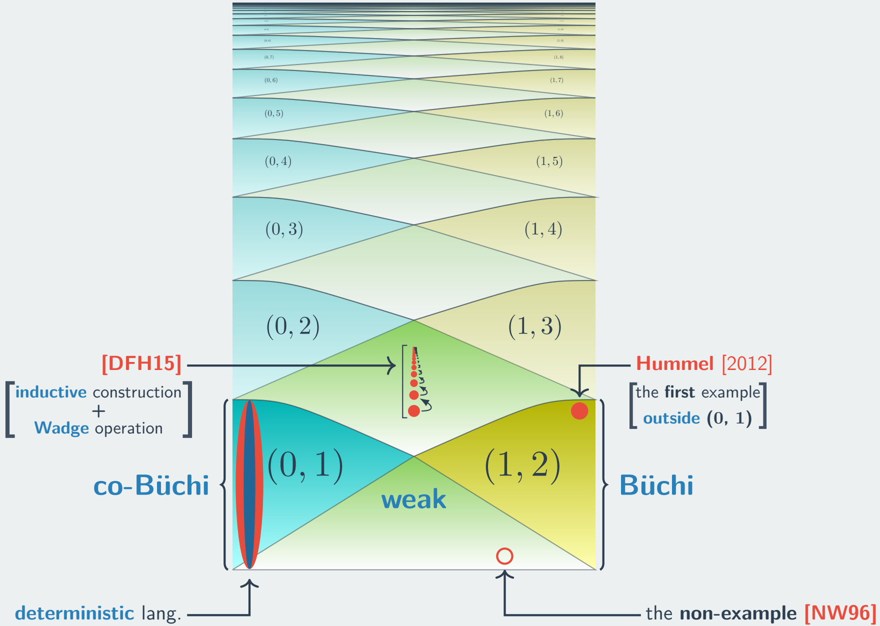

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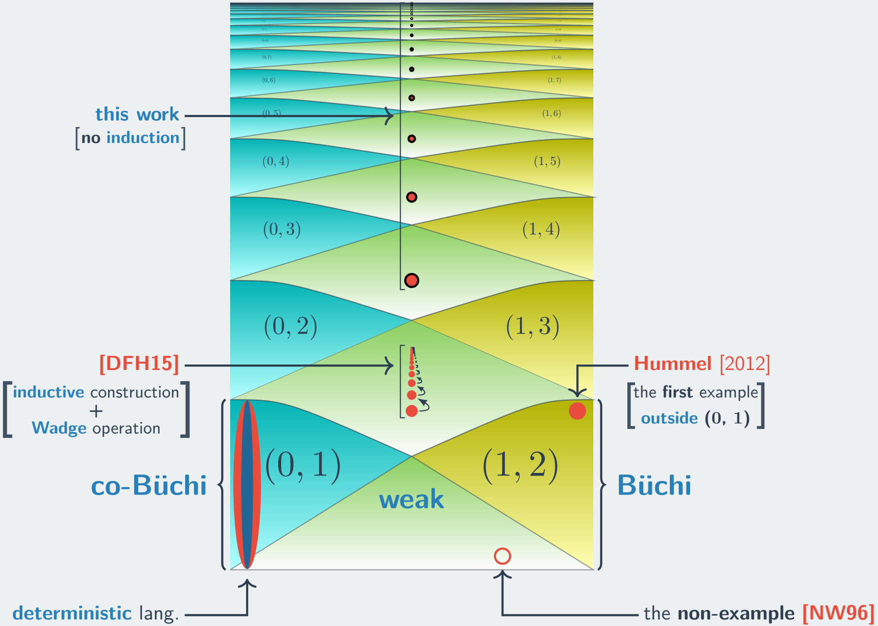

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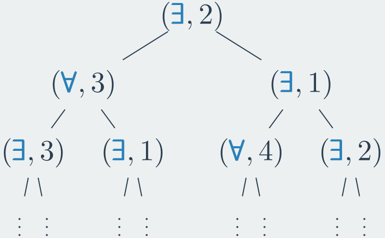

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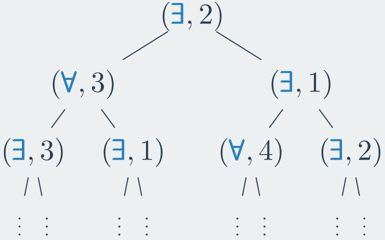

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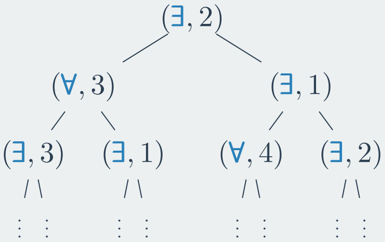

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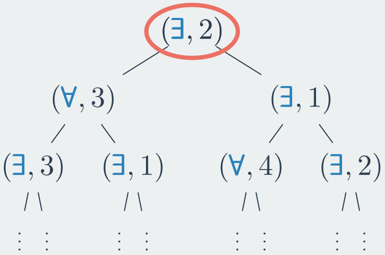

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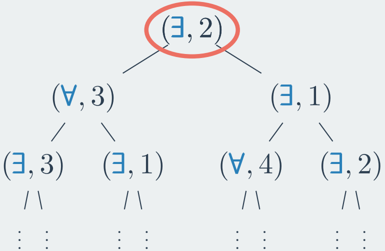

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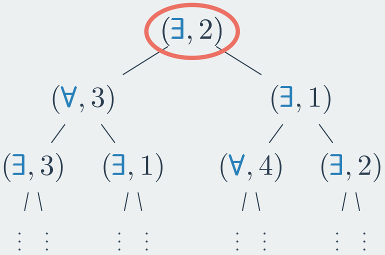

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$\exists \colon d_0 \quad (= \texttt{L})$

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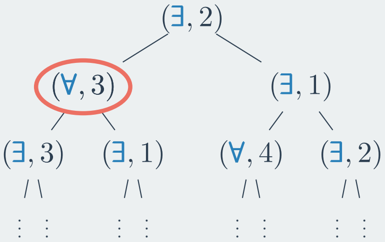

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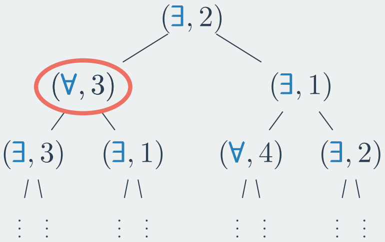

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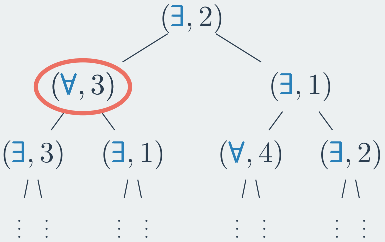

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$\forall$: d_1 $(= \mathtt{R})$

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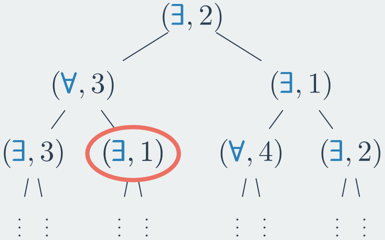

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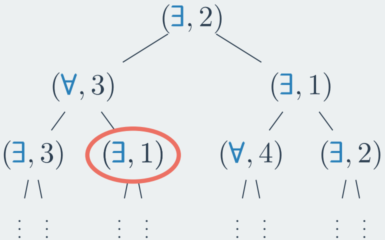

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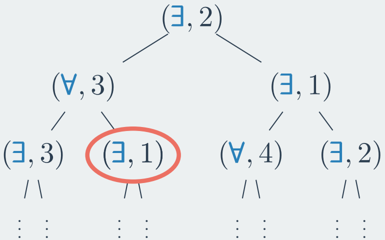

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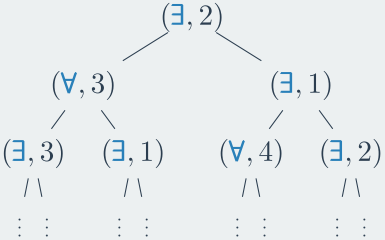

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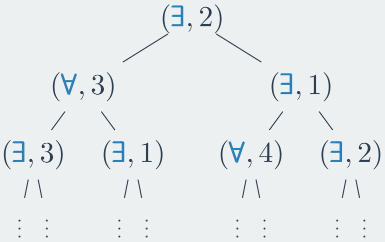

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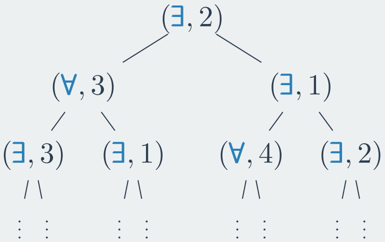

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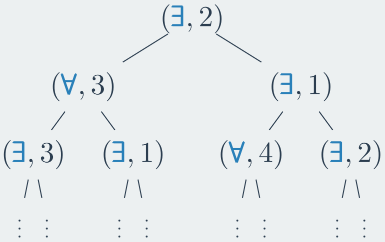

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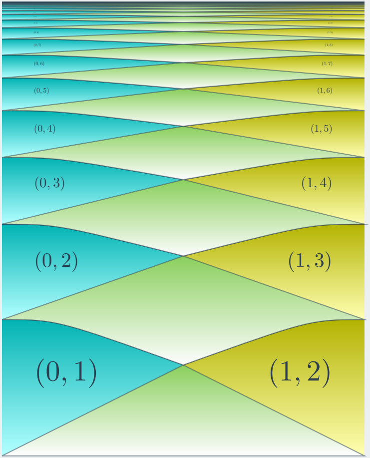
$t \in W_{i,j}$ **iff** $\exists$ has a winning strategy in $\mathcal{G}_t$

Hardness of the languages $W_{i,j}$



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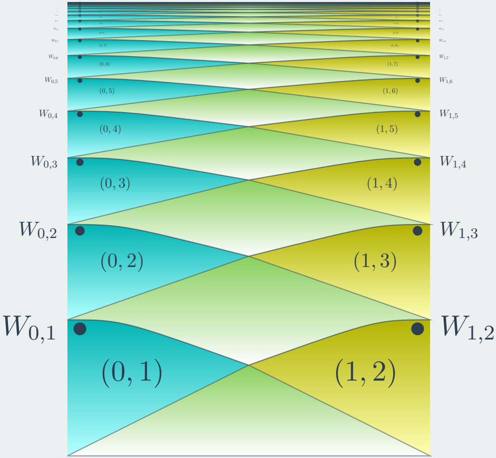

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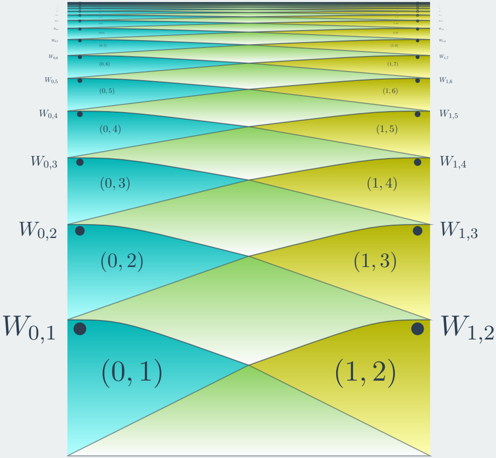

Theorem (Arnold [1999], Arnold, Niwiński [2007])

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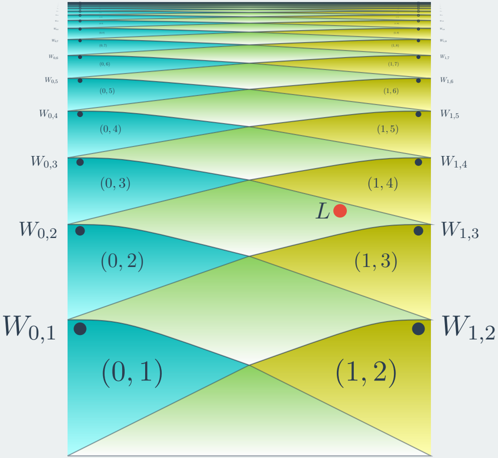

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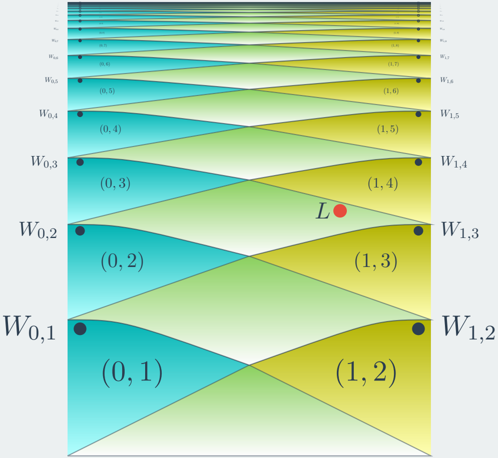

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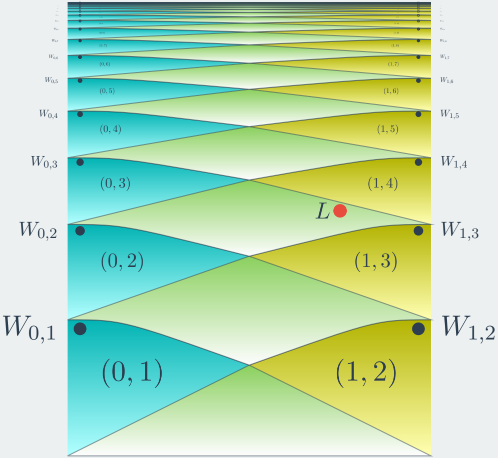

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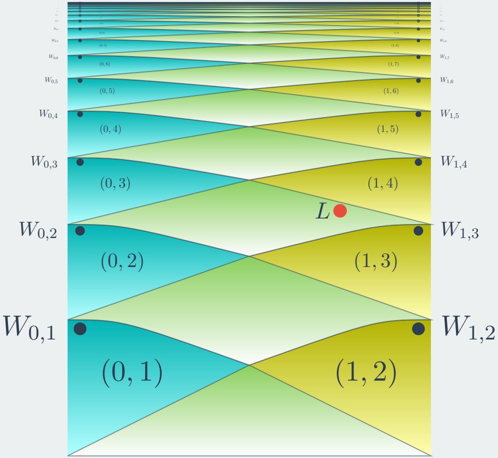

Theorem (Arnold [1999], Arnold, Niwiński [2007])

$$\underbrace{L \succ W_{i,j}}_{\text{Wadge order}} \implies L \text{ is } \textbf{not } (i,j) \text{ recognisable}$$

Idea

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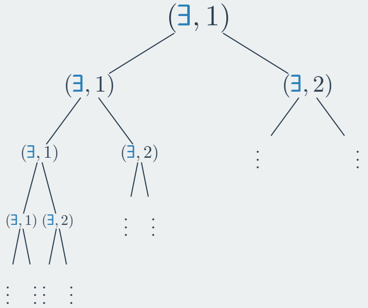

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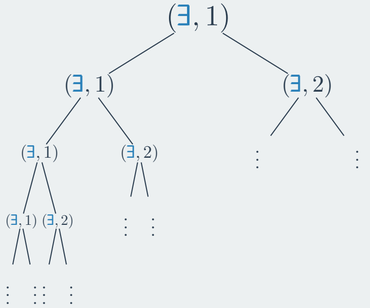

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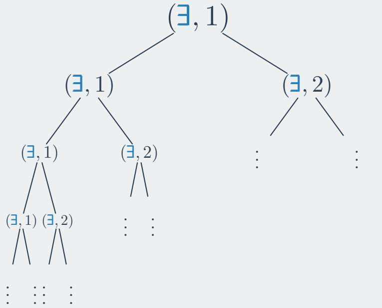

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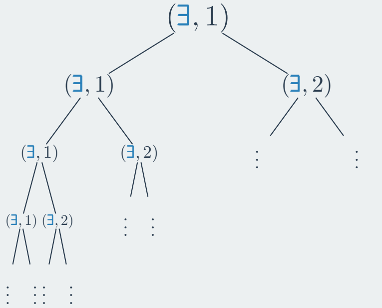
Staying within a winning region is **not** enough!

⇝ choose the **optimal** subtree

⇝ appraise the subtrees by their **signatures**



Signatures

Signatures [$(0,2)$ case]

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If $t \in W_{0,2}$ then $S_\exists^{\min}$ is winning.

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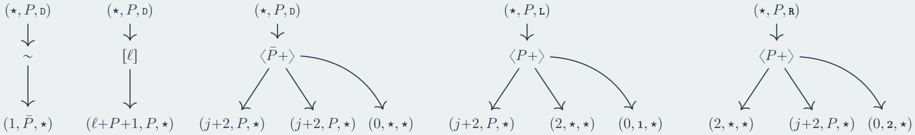

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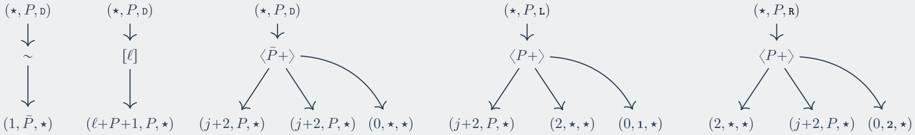

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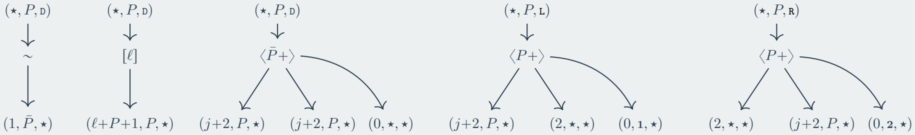

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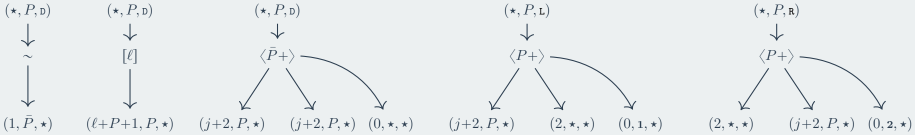

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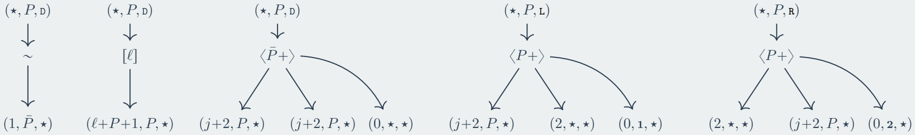

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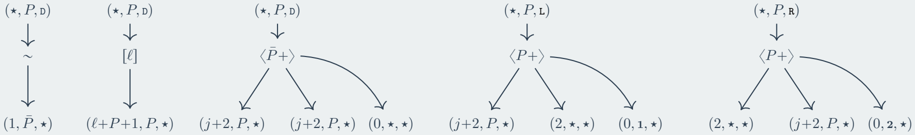

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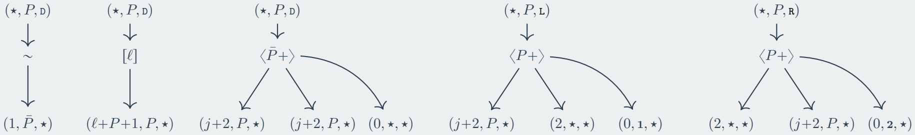

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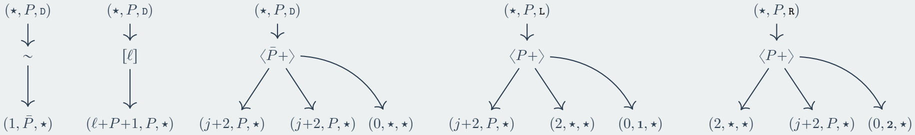

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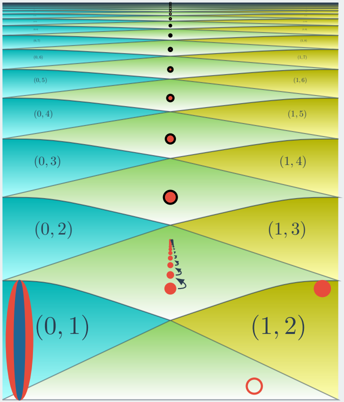

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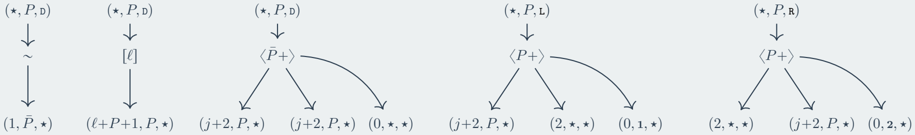

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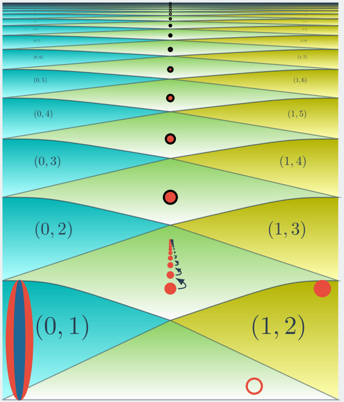

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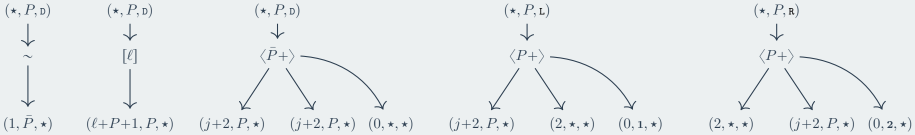

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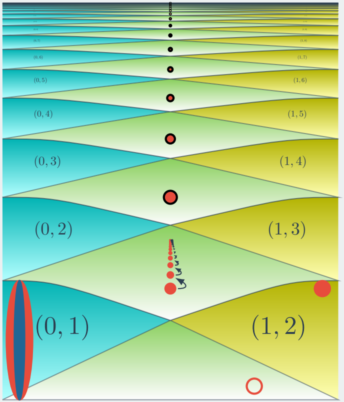

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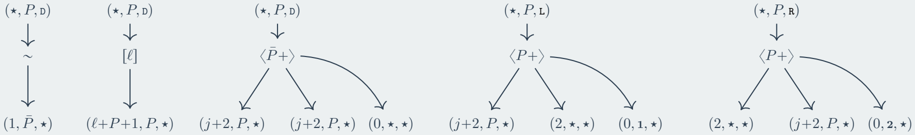

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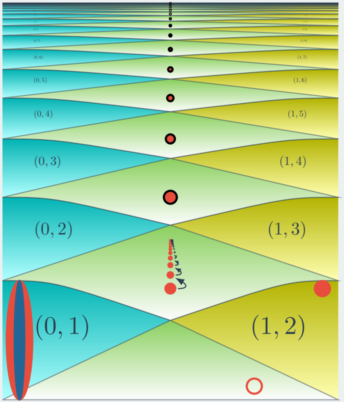

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