Büchi VASS recognise $\boldsymbol{\Sigma}_1^1$-complete $\boldsymbol{\omega}$-languages*

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Abstract. This paper exhibits an example of a $\boldsymbol{\Sigma}_1^1$-complete ω-language that can be recognised by a Büchi automaton with one partially blind counter (or equivalently a Büchi VASS with only one place). It follows as a corollary that there is no equivalent model of deterministic automata, even if we allow much richer data structures than just counters. The same holds for weaker forms of determinism, like for unambiguous or countably-unambiguous machines. This shows that even in the one counter case, non-determinism of Büchi VASS is inherent.

Keywords: Petri nets · infinite words · non-determinism.

In this work we study the strength of non-determinism in the context of partially blind multi-counter Büchi automata. This is a model of finite automata over infinite words with the Büchi acceptance condition (also known as "repeated reachability condition"). Additionally, each such automaton is equipped with a finite set of counters taking non-negative integer values. The automaton can freely increment and decrement the values of the counters, however it cannot test these values (i.e. no zero nor equality test). The only way in which the values of the counters influence the behaviour of the automaton is that they must stay non-negative during a run. The studied class of automata is strongly connected with other models based on Petri nets: a partially blind multi-counter Büchi automaton can be seen as a Büchi Vector Addition System with States (i.e. Büchi VASS) and vice versa.

Similarly as in the case of Petri nets, the considered model is naturally equipped with non-determinism. The main result of [10] implies that Büchi VASS are able to recognise ω-languages that cannot be recognised by the deterministic variant of the machines. This was achieved by topological methods: the paper provides an example of a Büchi VASS recognising an ω-language complete for the third level of the hierarchy of Borel sets ($\boldsymbol{\Sigma}_3^0$-complete); while deterministic Büchi VASS can only recognise ω-languages in the second level of the hierarchy (in $\boldsymbol{\Pi}_2^0$).

While the result of [10] separates non-deterministic Büchi VASS from the deterministic ones, it does not settle the question of the upper bounds on the topological complexity for these machines. Moreover, the lower bound of $\boldsymbol{\Sigma}_3^0$

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does not rule out the possibility of having a model of automata with a limited form of non-determinism that still captures the expressive power of non-deterministic Büchi VASS. To counterbalance the lack of full non-determinism, one could consider adding new counter operations (like min and max, see e.g. [1,3]); extending the acceptance condition to a topologically harder one, like Rabin, lim inf-parity, or something like the ωBS condition from [2]; or adding a richer data structure, e.g. a stack. Also, instead of a fully deterministic model, one could hope for an intermediate form of non-determinism, as in the case of unambiguous machines [5]; or when the non-deterministic choices appear only finitely many times in the accepting runs. The latter assumption implies that there are at most countably many accepting runs over a fixed ω-word, we will call such a machine *countably-unambigous*. This last restriction finds justification in the actual example provided in [10], where the whole non-deterministic choice of the machine reduces to choosing a single natural number at the beginning of a run.

In general, topological complexity suits well to make a distinction between determinism and non-determinism. Firstly, in the case of all standard models of machines, the relation $\mathrm{run}(\alpha,\rho)$ of "being a run" is a closed[1] relation between ω-words $\alpha \in A^\omega$ and sequences of configurations $\rho \in C^\omega$. Moreover, for all the standard acceptance conditions mentioned above, the property of being an accepting run $\mathrm{acc}(\rho)$ is Borel. This implies that all deterministic devices, which can be seen as transducers of an input ω-word into a sequence of configurations, recognise only Borel sets. The situation is different in the case of non-deterministic devices, where the language of such a machine can be written as a projection of a Borel set:

$$\{\alpha \in A^\omega \mid \exists \rho \in C^\omega.\, \mathrm{run}(\alpha,\rho) \wedge \mathrm{acc}(\rho)\} = \pi_{A^\omega}\big(\{(\alpha,\rho) \mid \mathrm{run}(\alpha,\rho) \wedge \mathrm{acc}(\rho)\}\big). \quad (1)$$

It is known that in general, projections of Borel sets might not be Borel — they form a wider class of *analytic* sets (denoted $\mathbf{\Sigma}^1_1$). Thus, $\mathbf{\Sigma}^1_1$ is the upper bound for the topological complexity of general non-deterministic devices. The above formula, together with a theorem by Lusin and Novikov [12, Theorem 18.10], imply that countably-unambiguous machines recognise only Borel ω-languages. This means that in terms of topological complexity they are closer to deterministic than to non-deterministic ones.

The above topological results say that the distinction between weak vs. full forms of non-determinism can be topologically understood as the difference between Borel and analytic sets. The purpose of the present paper is to use this correspondence by showing the following theorem.

Theorem 1. *There exists an ω-language that is recognised by a Büchi VASS with one counter (i.e. with one place) that recognises a $\mathbf{\Sigma}^1_1$-complete ω-language.*

As noted above, all ω-languages recognised by non-deterministic Büchi VASS are in $\mathbf{\Sigma}^1_1$. Thus, the above result solves the question of the upper bounds for the topological complexity of these machines. Moreover, the theorem translates to the automata theoretic realms as the following corollary.

[1] Equivalently: a relation given by a *safety* condition.

Corollary 1. *No model of deterministic, unambiguous, nor even countably-unambiguous automata with countably many configurations and a Borel acceptance condition can capture the class of ω-languages recognisable by Büchi VASS with one counter.*

The crucial difficulty in proving Theorem 1 is the fact, that Büchi VASS are partially blind: they cannot test their counters for exact values. As a consequence, there is a natural simulation order on the configurations of a Büchi VASS: a configuration $(q, \vec{a})$ *simulates* $(q, \vec{c})$ if they have the same state and the counter values $\vec{a}$ and $\vec{c}$ satisfy coordinate-wise $\vec{a} \geq \vec{c}$. In such a case, the language recognised from $(q, \vec{a})$ contains the language recognised by $(q, \vec{c})$; because each accepting run from $(q\,\vec{c})$ can be lifted to an accepting run from $(q, \vec{a})$ just by increasing the counter values. In particular, when there is exactly one counter, the maximal size of an anti-chain of the simulation order is bounded by the number of states; what limits the possible structure of the so-called *residual* ω-languages of the device.

Although the construction of the paper is expressed in terms of topological complexity, the actual core of the proof is a combinatorial idea allowing to simulate a $\mathbf{\Sigma}^1_1$-hard behaviour (i.e. one that involves full non-determinism) by an efficient way of storing information in the value of a unique partially blind counter of the automaton. The idea is not very complex, and the overall construction should be considered as rather direct.

To simplify the presentation of the proof it is performed in three steps. In Section 2 we provide an easy example of a $\mathbf{\Sigma}^1_1$-complete ω-language recognised by a Büchi VASS with two counters. Then, in Section 3 we characterise a specific $\mathbf{\Sigma}^1_1$-complete set (namely $\mathrm{IF}_{\mathrm{inf}}$). This set is in a certain sense monotone, which is used to reflect the simulation order on configurations of our automata. In Section 4 we reduce the set $\mathrm{IF}_{\mathrm{inf}}$ to an ω-language recognised by a Büchi VASS with only one counter, which concludes the proof of Theorem 1. Section 5 is devoted to Corollary 1. Finally, Section 6 gives some concluding remarks.

Acceptance condition The results of the paper speak about VASS with the Büchi acceptance condition. Since non-deterministic Büchi automata recognise all ω-regular languages, these machines can simulate all other ω-regular acceptance conditions. Thus, the Büchi condition seems to be one of the canonical ones (with most of them actually equivalent). On the other hand, the situation is different for certain weaker acceptance conditions: the safety, reachability, and co-Büchi conditions can be written as countable unions of closed sets (i.e. $\mathbf{\Sigma}^0_2$). A known topological fact says that a projection of a $\mathbf{\Sigma}^0_2$ set contained in a compact[2] topological space is also $\mathbf{\Sigma}^0_2$. Therefore, none of these weaker conditions allows a non-deterministic VASS to recognise a non-Borel ω-language. A reasonable task (although out of the scope of the present paper) is to design deter-

[2] The space of runs is compact because the automata do not admit ϵ-transitions and therefore the possible counter values are bounded at each fixed place of the input ω-word.

ministic or almost deterministic models for VASS with these weaker acceptance conditions.

Related work There is a number of papers studying the topological complexity of sets recognisable by various models of machines [13,6,8,4,7]. In certain cases, the topological lower bounds were used to separate models of machines [1,10]. Also, high topological complexity of some classes of languages can influence their decidability [11].

The question of upper bounds on the topological complexity for Büchi VASS was left as an open problem in [10]. After publication of that article, the authors independently managed to solve this problem. In [9], Finkel has found a family of Büchi VASS with four counters that recognise ω-languages at all Wadge degrees of non-deterministic Turing machines. This result implies that there are Büchi VASS with 4 counters recognising $\mathbf{\Sigma}^1_1$-complete ω-languages. Moreover, it shows that many intermediate classes of topological complexity are also inhabited by such ω-languages. However, it is not clear whether the number of counters in that construction can be reduced. This paper provides a construction of a single $\mathbf{\Sigma}^1_1$-complete ω-language recognised using only one counter. Thus, the two results are mathematically incomparable.

1 Preliminary notions

We use $\omega = \{0, 1, \ldots\}$ to denote the set of natural numbers. If A is a non-empty set then A^* and A^ω are respectively sets of finite and infinite sequences of elements of A. The elements of A^* are called *words* and the elements of A^ω are called ω-words. An *ω-language* is a set of ω-words. If $v \in A^*$ then by $|v| \in \omega$ we denote the length of v (i.e. the number of symbols in v). By $v \cdot x$ we denote the concatenation of the two sequences, with $|v \cdot x| = |v| + |x|$. If the context is clear, we skip the concatenation symbol $\cdot$. If $n \leq |v|$ then by $v{\restriction}_n \in A^n$ we denote the restriction of the sequence to its first n symbols.

Büchi VASS A Büchi VASS (or shortly VASS, as we consider only the Büchi acceptance condition) is a tuple $\mathcal{A} = \langle A, Q, q_\mathrm{I}, F, C, \delta\rangle$, where:

- A is a finite *input alphabet*,
- Q is a finite set of *states*,
- $q_\mathrm{I} \in Q$ is an *initial state*,
- $F \subseteq Q$ is a set of *accepting states*,
- C is a finite set of *counters*,
- δ is a finite *transition relation*, its elements are *transitions* (q, a, τ, q') where $q, q' \in Q$, $a \in A$, and $\tau \colon C \to \mathbb{Z}$.

Without loss of generality we assume that the set of counters C has the form $C = \{1, 2, \ldots, k\}$ for some k (in this work 1 or 2). We visually represent a transition (q, a, τ, q') by $q \xrightarrow{a:\left(\tau(1),\tau(2),\ldots,\tau(k)\right)} q'$. We say that such a transition is *over* the letter a. If $A' \subseteq A$ then $q \xrightarrow{A' \colon \left(\tau(1), \tau(2), \ldots, \tau(k)\right)} q'$ means that for each $a \in A'$ there is a respective transition. Similarly, $q \xrightarrow{a} q'$ and $q \xrightarrow{A'} q'$ denote the respective transitions that do not modify the counter values (i.e. τ is constant 0).

A *configuration* of a VASS $\mathcal{A}$ is a tuple $(q, c_1, c_2, \ldots, c_k)$ where $q \in Q$, $c_1, \ldots, c_k \in \omega$, and $\{1, \ldots, k\} = C$. The *initial configuration* is $(q_{\mathrm{I}}, 0, \ldots, 0)$. We say that a transition $q \xrightarrow{a \colon \left(\tau(1), \ldots, \tau(k)\right)} q'$ *goes* from a configuration $(q, c_1, \ldots, c_k)$ to a configuration $\left(q', c_1 + \tau(1), \ldots, c_k + \tau(k)\right)$ (note that by the definition it requires all the numbers $c_i + \tau(i)$ to be non-negative).

Let $\alpha \in A^\omega$ be an ω-word over the input alphabet. A *run* of a VASS $\mathcal{A}$ over α is an infinite sequence ρ of configurations, such that $\rho(0)$ is the initial configuration and for every $i \in \omega$ there is a transition of $\mathcal{A}$ over the letter $\alpha(i)$ that goes from the configuration $\rho(i)$ to the configuration $\rho(i{+}1)$. A run ρ is *accepting* if for infinitely many i the configuration $\rho(i) = (q_i, \ldots)$ satisfies $q_i \in F$ (i.e. it visits infinitely many times an accepting state). A VASS $\mathcal{A}$ *accepts* an ω-word α if there exists an accepting run of $\mathcal{A}$ over α. The *language* of $\mathcal{A}$ (denoted $\mathrm{L}(\mathcal{A})$) is the set of ω-words accepted by $\mathcal{A}$.

Topology We will use the standard notions of topology on Polish spaces [12]. The space A^ω of all ω-words over a finite alphabet A can be naturally endowed with a topology where open sets are those obtained as unions of *basic open* sets of the form $N_u \stackrel{\text{def}}{=} \{u{\cdot}\alpha \mid \alpha \in A^\omega\}$. A set whose complement is open is called *closed*. Closed subsets C of A^ω can be equivalently characterised as those satisfying the following *safety* property:

$$\forall \alpha \in A^\omega.\ \left(\forall n \in \omega.\ \exists \beta \in A^\omega.\ \alpha{\restriction}_n \cdot \beta \in C\right) \Longrightarrow \alpha \in C. \tag{2}$$

The family of Borel sets in a topological space X is the smallest σ-algebra that contains all the open sets in X. By $\mathbf{\Sigma}_1^1$ we denote the family of analytic sets, i.e. projections of Borel sets. A function $f \colon X \to Y$ between two topological spaces is *continuous* if the pre-image $f^{-1}(U) \subseteq X$ is open for every open[3] set $U \subseteq Y$. If $A \subseteq X$ and $B \subseteq Y$ are two subsets of topological spaces then we call $f \colon X \to Y$ a *reduction* of A to B if $f^{-1}(B) = A$. If Γ is a class of sets and $G \subseteq X$ is a subset of a topological space X, we say that G is *Γ-hard* if for every set $A \in \Gamma$ there exists a continuous reduction of A to G. If additionally $G \in \Gamma$ then we say that G is *Γ-complete*. Since continuous reductions can be composed, we obtain the following fact.

Fact 2. *If G is Γ-hard and G continuously reduces to G' then also G' is Γ-hard.*

Orders Consider a set X and a relation $o \subseteq X \times X$ on X. We say that o is a *linear order* if it is reflexive, transitive, and anti-symmetric. We interpret a pair $(x, x') \in o$ as representing the fact that x is o-smaller-or-equal than x'. A linear order o is *ill-founded* if there exists an infinite sequence $x_0, x_1, \ldots$ of pairwise

[3] Since $f^{-1}\left(\bigcup \mathcal{F}\right) = \bigcup f^{-1}\left(\mathcal{F}\right)$, it is enough to consider basic open sets U.

distinct elements of X such that for all n we have $(x_{n+1}, x_n) \in o$. Such a sequence indicates an infinite o-descending chain. An order that is not ill-founded is called *well-founded*.

Binary trees The binary tree is the set of all sequences of *directions* $\mathcal{T} \stackrel{\text{def}}{=} \{\mathtt{L}, \mathtt{R}\}^*$ where the *directions* $\mathtt{L}$, $\mathtt{R}$ are two fixed distinct symbols. For technical reasons we sometimes consider a third direction $\mathtt{M}$ (it does not occur in the binary tree).

A set $X \subseteq \mathcal{T}$ can be naturally identified with its characteristic function $X \in \{0,1\}^{\left(\{\mathtt{L},\mathtt{R}\}^*\right)}$. Thus, the family of all subsets of the binary tree, with the natural product topology, is homeomorphic with the Cantor set $\{0,1\}^\omega$.

The elements $v, x \in \mathcal{T}$ are called *nodes*. Nodes are naturally ordered by the following three orders:

- the prefix order: $v \preceq x$ if x can be obtained by concatenating something at the end of v,
- the lexicographic order: $v \leq_{\mathrm{lex}} x$ if v is lexicographically smaller than x (we assume that $\mathtt{L} <_{\mathrm{lex}} \mathtt{M} <_{\mathrm{lex}} \mathtt{R}$),
- the infix order: $v \leq_{\mathrm{inf}} x$ if $v\mathtt{M}^\omega$ (i.e. the ω-word obtained by appending infinitely many symbols $\mathtt{M}$ after v) is lexicographically less or equal than $x\mathtt{M}^\omega$.

Notice that, for every fixed n, when restricted to $\{\mathtt{L}, \mathtt{R}\}^n$, the lexicographic and infix orders coincide. However, $\mathtt{L} <_{\mathrm{inf}} \epsilon <_{\mathrm{inf}} \mathtt{R}$ but ϵ is the minimal element of $\leq_{\mathrm{lex}}$. Both the lexicographic and infix orders are linear.

Since the infix order is countable, dense, and has no minimal nor maximal elements, we obtain the following fact.

Fact 3. $(\mathcal{T}, \leq_{\mathrm{inf}})$ *is isomorphic to the order of rational numbers* $(\mathbb{Q}, \leq)$.

Hardness In the following part of the paper we will use the following two sets:

$$\mathrm{IF}_{\mathrm{pre}} \stackrel{\text{def}}{=} \{X \subseteq \mathcal{T} \mid X \text{ contains an infinite } \preceq\text{-ascending chain}\},$$

$$\mathrm{IF}_{\mathrm{inf}} \stackrel{\text{def}}{=} \{X \subseteq \mathcal{T} \mid X \text{ contains an infinite } \leq_{\mathrm{inf}}\text{-descending chain}\}.$$

The following lemma is a standard topological observation.

Lemma 1. *The sets* $\mathrm{IF}_{\mathrm{pre}}$ *and* $\mathrm{IF}_{\mathrm{inf}}$ *are* $\boldsymbol{\Sigma}^1_1$*-complete.*

Proof. Both sets belong to $\boldsymbol{\Sigma}^1_1$ just by the form of the definition. $\mathrm{IF}_{\mathrm{pre}}$ is $\boldsymbol{\Sigma}^1_1$-hard by an easy reduction from the set of ill-founded ω-branching trees, the proof is similar to [12, Exercise 27.3].

$\mathrm{IF}_{\mathrm{inf}}$ is $\boldsymbol{\Sigma}^1_1$-hard by a reduction from the set of ill-founded linear orders on ω (seen as elements of $\{0,1\}^{\omega\times\omega}$). Let us prove this fact more formally. Consider an element $o \in \{0,1\}^{\omega\times\omega}$ that is a linear order on ω. The latter set is $\boldsymbol{\Sigma}^1_1$-complete by a theorem by Lusin and Sierpiński [12, Theorem 27.12]. We will inductively define $X_o \subseteq \mathcal{T}$ in such a way to ensure that $o \mapsto X_o$ is a continuous mapping and o is ill-founded if and only if $X_o \in \mathrm{IF}_{\mathrm{inf}}$.

Let us proceed inductively, defining a sequence of nodes $(x_n)_{n\in\omega} \subseteq \mathcal{T}$. Our invariant says that $|x_k| = k$ and the map $k \mapsto x_k$ is an isomorphism of the orders $\big(\{0,1,\ldots,n\}, o\big)$ and $\big(\{x_0, x_1, \ldots, x_n\}, \leq_{\mathrm{inf}}\big)$. We start with $x_0 = \epsilon$ (i.e. the root of $\mathcal{T}$). Assume that $x_0, \ldots x_n$ are defined and satisfy the invariants. By the definition of $\leq_{\mathrm{inf}}$, there exists a node $x \in \{\mathtt{L}, \mathtt{R}\}^{n+1}$ such that for $k = 0, 1, \ldots, n$ we have $x \leq_{\mathrm{inf}} x_k$ if and only if $(n{+}1, k) \in o$. Let x_{n+1} be such a node.

The above induction defines an infinite sequence of nodes $x_0, x_1, \ldots$ Let $X_o \stackrel{\mathrm{def}}{=} \{x_n \mid n \in \omega\} \subseteq \mathcal{T}$. By the definition of X_o, the mapping $o \mapsto X_o$ is continuous — the fact whether a node $x \in \mathcal{T}$ belongs to X_o depends only on $o \cap \{0, 1, \ldots, |x|\}^2$. Using our invariant, we know that the map $k \mapsto x_k$ is an isomorphism of the orders $\big(\omega, o\big)$ and $\big(X_o, \leq_{\mathrm{inf}}\big)$. Thus, o is ill-founded if and only if $X_o \in \mathrm{IF}_{\mathrm{inf}}$. $\square$

2 Hardness for two counters

In this section we provide a simple example of an ω-language that is $\mathbf{\Sigma}^1_1$-complete and can be recognised by a VASS $\mathcal{A}_2$ with two counters. This example should be seen as a preliminary step towards the one counter example given in Section 4.

The VASS $\mathcal{A}_2$ is depicted in Figure 1. Let $A_0 \stackrel{\mathrm{def}}{=} \big\{<, d_1, d_2, |, i_1, i_2, +, -, >\big\}$ and let the alphabet $A \stackrel{\mathrm{def}}{=} A_0 \cup \{\sharp\}$. The initial state is q_0, the single accepting state is q_a. The only non-determinism occurs in q_0 when reading $<$ — the VASS can stay in q_0 or move to q_1. Only the states q_1 and q_2 modify the counter values.

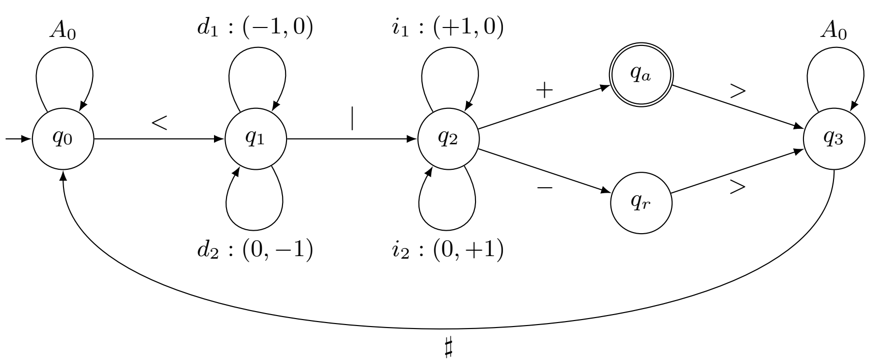

Fig. 1. The VASS $\mathcal{A}_2$ with two counters that recognises a $\mathbf{\Sigma}^1_1$-complete ω-language.

Lemma 2. *There exists a continuous reduction from* $\mathrm{IF}_{\mathrm{pre}}$ *to the ω-language recognised by $\mathcal{A}_2$.*

Intuition An ω-word accepted by $\mathcal{A}_2$ consists of infinitely many *phases* separated by $\sharp$. Each phase is a finite word over the alphabet A_0. In our reduction we will restrict to phases being sequences of *blocks*, each block being a finite word of the form given by the following definition (for $n_1, n_2, m_1, m_2 \in \omega$ and $s \in \{+, -\}$):

$$B^s(-n_1, -n_2, +m_1, +m_2) \stackrel{\text{def}}{=} < d_1^{n_1}\ d_2^{n_2}\ |\ i_1^{m_1}\ i_2^{m_2}\ s > \ \in A_0^*. \qquad (3)$$

Such a block is *accepting* if $s = +$, otherwise $s = -$ and the block is *rejecting*. If $\mathcal{A}_2$ starts reading a block and moves from q_0 to q_1 over $<$ then we say that it *chooses* this block. Otherwise $\mathcal{A}_2$ stays in q_0 and it does not *choose* the given block. By the construction of the VASS $\mathcal{A}_2$, in every run it needs to choose exactly one block from each phase. Additionally, the run is accepting if and only if infinitely many of the chosen blocks are accepting.

In our reduction we will represent a given set $X \subseteq \mathcal{T}$ by an appropriately defined sequence of phases. We will control the set of configurations the VASS can reach at the beginning of each phase. These configurations will form an anti-chain with respect to the coordinate-wise (or simulation) order: if the VASS can reach two distinct configurations (q_0, c_1, c_2) and (q_0, c_1', c_2') then either $c_1 < c_1'$ and $c_2 > c_2'$; or $c_1 > c_1'$ and $c_2 < c_2'$. Each block in the successive phase will be of the form $B^s(-c_1, -c_2, +m_1, +m_2)$ for some reachable configuration (q_0, c_1, c_2) — this will be the only reachable configuration in which the automaton can choose the considered block. After choosing it, the automaton will finish reading the phase in the configuration (q_3, m_1, m_2).

Proof of Lemma 2 For the rest of this section we prove Lemma 2. Let us fix a set $X \subseteq \mathcal{T}$. We will construct an ω-word $\alpha(X) \in A^\omega$. The ω-word $\alpha(X)$ will consist of infinitely many phases $\alpha(X) = u_0 \sharp u_1 \sharp \cdots$, for $u_n \in A_0^*$. The n-th phase u_n (for $n = 0, 1, \ldots$) will depend on $X \cap \{\mathtt{L}, \mathtt{R}\}^n$. This will guarantee that the function $\alpha\colon 2^{\mathcal{T}} \to A^\omega$ is continuous. The proof will be concluded by the following claim.

Claim. X has an infinite $\preceq$-ascending chain if and only if $\mathcal{A}_2$ accepts $\alpha(X)$.

To simplify the construction, let us define inductively the function $b\colon \mathcal{T} \to \omega$, assigning to nodes $v \in \mathcal{T}$ their binary value $b(v)$:

- $b(\epsilon) = 0$,
- $b(v\mathtt{L}) = 2 \cdot b(v)$,
- $b(v\mathtt{R}) = 2 \cdot b(v) + 1$.

Let $b'(v) = 2^n - b(v) - 1$ for $n = |v|$ (i.e. $v \in \{\mathtt{L}, \mathtt{R}\}^n$). Note that for every $n \in \omega$ we have

$$b\big(\{\mathtt{L}, \mathtt{R}\}^n\big) = b'\big(\{\mathtt{L}, \mathtt{R}\}^n\big) = \{0, 1, \ldots, 2^n - 1\},$$

and both b and b' are bijective between these sets. Additionally, if $v \neq v' \in \{\mathtt{L}, \mathtt{R}\}^n$ then either $b(v) < b(v')$ and $b'(v) > b'(v')$; or $b(v) > b(v')$ and $b'(v) < b'(v')$.

We take any $n = 0, 1, \ldots$ and define the n-th phase u_n. Let u_n be the concatenation of the following blocks, for all $v \in \{\mathtt{L}, \mathtt{R}\}^n$ and $d \in \{\mathtt{L}, \mathtt{R}\}$:

$$B^s\big(-b(v), -b'(v), +b(vd), +b'(vd)\big),$$

where $s = +$ if $v \in X$ and $s = -$ otherwise. Thus, the n-th phase is a concatenation of 2^{n+1} blocks, one for each node vd in $\{\mathtt{L}, \mathtt{R}\}^{n+1}$.

To prove Claim 2 it is enough to notice the following fact.

Fact 4. *There is a bijection between infinite branches $\beta \in \{\mathtt{L}, \mathtt{R}\}^\omega$ and runs ρ of $\mathcal{A}_2$ over $\alpha(X)$. The bijection satisfies that the configuration in ρ before reading the n-th phase of $\alpha(X)$ is $\big(q_0, b(v_n), b'(v_n)\big)$ for $v_n = \beta{\restriction}_n \in \{\mathtt{L}, \mathtt{R}\}^n$. $\mathcal{A}_2$ visits an accepting state in ρ while reading the n-th phase of $\alpha(X)$ if and only if $v_n \in X$.*

Proof. Easy induction. □

This concludes the proof of Lemma 2.

3 Representation of $\mathbf{IF}_{\mathbf{pre}}$

To construct our continuous reduction in the one-counter case, we need the following simple lemma that provides an alternative characterisation of the set $\mathrm{IF}_{\mathrm{inf}}$. Let us introduce the following definition.

Definition 1. *A sequence $v_0, v_1 \ldots \in \mathcal{T}$ is called a* correct chain *if $v_0 = \epsilon$ and for every $n = 0, 1, \ldots$:*

1. *$|v_{n+1}| = |v_n| + 1$,*
2. *$v_{n+1} \leq_{\mathrm{inf}} v_n\mathtt{R}$ (or equivalently $v_{n+1} \leq_{\mathrm{lex}} v_n\mathtt{R}$).*

A correct chain is witnessing *for a set $X \subseteq \mathcal{T}$ if for infinitely many n we have $v_n \in X$ and $v_{n+1} \leq_{\mathrm{inf}} v_n\mathtt{L}$.*

Intuitively, the definition forces the sequence to be *not so-much increasing* in the infix order $\leq_{\mathrm{inf}}$: the successive element v_{n+1} needs to be *to the left* in the tree from $v_n\mathtt{R}$. Such a sequence is *witnessing* for a set X if infinitely many times it belongs to X and at these moments it actually drops in $\leq_{\mathrm{inf}}$.

Lemma 3. *A set $X \subseteq \mathcal{T}$ belongs to* $\mathrm{IF}_{\mathrm{inf}}$ *if and only if there exists a correct chain witnessing for X.*

Proof. First take a correct chain witnessing for X. Let $x_0, x_1, \ldots$ be the subsequence that shows that $(v_n)_{n\in\omega}$ is witnessing for X. In that case, by the definition, for all n we have $x_n \in X$ and $x_{n+1} <_{\mathrm{inf}} x_n$ (because $x_{n+1}\mathtt{M}^\omega \leq_{\mathrm{lex}} x_n\mathtt{L}\mathtt{R}^\omega <_{\mathrm{lex}} x_n\mathtt{M}^\omega$). Thus, X has an infinite $\leq_{\mathrm{inf}}$-descending chain and belongs to $\mathrm{IF}_{\mathrm{inf}}$.

Now assume that $X \in \mathrm{IF}_{\mathrm{inf}}$ and $x_0 >_{\mathrm{inf}} x_1 >_{\mathrm{inf}} x_2 >_{\mathrm{inf}} \ldots$ is a sequence witnessing that. Without loss of generality we can assume that $|x_{n+1}| > |x_n|$ because for each fixed depth k there are only finitely many nodes of $\mathcal{T}$ in $\{\mathtt{L}, \mathtt{R}\}^{\leq k}$. We can now add intermediate nodes in-between the sequence $(x_n)_{n\in\omega}$ to construct a correct chain witnessing for X; the following pseudo-code realises this goal:

```
n := 0;
i := 0;
while (true) {
  if (n > |x_i|) {
    i := i+1;
  }

  v_n := x_i↾_n;
  n := n+1;
}
```

Clearly, Property 1 in the definition of a correct chain is guaranteed. Let $i \in \omega$ and $n = |x_i|$. By the fact that $x_{i+1} <_{\mathrm{inf}} x_i$ we know that $x_{i+1}\restriction_{n+1} \leq_{\mathrm{inf}} x_i\mathtt{L}$. Therefore, for every $n \in \omega$ we have $v_{n+1} \leq_{\mathrm{inf}} v_n\mathtt{R}$ and if $n = |x_i|$ for some i then $v_{n+1} \leq_{\mathrm{inf}} v_n\mathtt{L}$. It implies that the sequence $(v_n)_{n\in\omega}$ satisfies Property 2 in the definition of a correct chain. It is clearly witnessing for X because it contains $(x_n)_{n\in\omega}$ as a subsequence. □

4 Hardness for one counter

In this section we provide an example of an ω-language that is $\mathbf{\Sigma}^1_1$-complete and can be recognised by a VASS $\mathcal{A}_1$ with one counter. $\mathcal{A}_1$ is depicted in Figure 2, it is very similar to $\mathcal{A}_2$, but simpler. Let $A_0 \stackrel{\text{def}}{=} \{<, d, |, i, +, -, >\}$ and let the alphabet $A \stackrel{\text{def}}{=} A_0 \cup \{\sharp\}$.

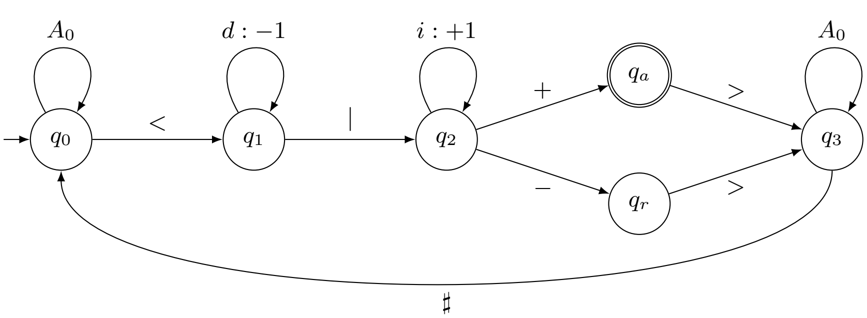

Fig. 2. The VASS $\mathcal{A}_1$ with one counter that recognises a $\mathbf{\Sigma}^1_1$-complete ω-language.

Proposition 1. *There exists a continuous reduction from* $\mathrm{IF}_{\mathrm{inf}}$ *to the* ω*-language recognised by* $\mathcal{A}_1$.

Similarly as before, we will use the notion of phases and blocks. Since there is only one counter now (and only two letters modifying its value: d and i) we exchange the definition of a block (see (3)) by the following one (for $n, m \in \omega$ and $s \in \{+, -\}$):

$$B^s(-n, +m) \stackrel{\text{def}}{=} < d^n \mid i^m \ s > \ \in A_0^*. \tag{4}$$

Similarly as before, we will take a set $X \subseteq \mathcal{T}$ and construct an ω-word $\alpha(X)$. This ω-word will be a concatenation of infinitely many phases $u_0 \sharp u_1 \sharp \cdots$. The n-th phase u_n will depend on $X \cap \{\mathtt{L}, \mathtt{R}\}^n$. The configurations (q_0, c) reached at the beginning of an n-th phase will be in correspondence with the nodes $v \in \{\mathtt{L}, \mathtt{R}\}^n$. The bigger the value c, the higher in the infix order (or the lexicographic order, as they overlap here) the respective node v is.

To precisely define our ω-word $\alpha(X)$ we need to define fast-growing functions: $m\colon \{-1\} \cup \omega \to \omega$ and $e\colon \mathcal{T} \to \omega$:

$$\begin{aligned} m(-1) &= 1, && \\ m(n) &= m(n-1) \cdot 2^n && \text{for } n \in \omega, \\ e(v) &= m(|v|-1) \cdot b(v) && \text{for } v \in \mathcal{T}. \end{aligned}$$

These functions allow to use a big range of the possible values of a single counter of a VASS to represent particular nodes of the tree. We will use the following two invariants of this definition, for $n \in \omega$ and $v, v' \in \{\mathtt{L}, \mathtt{R}\}^n$:

$$v <_{\text{inf}} v' \iff e(v) \le e(v'), \tag{5}$$

$$e(v) + m(|v|-1) \ \le \ m(|v|). \tag{6}$$

We take any $n = 0, 1, \ldots$ and define the n-th phase u_n. Let u_n be the concatenation of the following blocks, for all $v \in \{\mathtt{L}, \mathtt{R}\}^n$ and $d \in \{\mathtt{L}, \mathtt{R}\}$:

$$B^s\big(-e(v), +e(vd)\big),$$

where $s = +$ if $v \in X$ and $d = \mathtt{L}$; otherwise $s = -$. Thus, the n-th phase is a concatenation of 2^{n+1} blocks, one for each node vd in $\{\mathtt{L}, \mathtt{R}\}^{n+1}$.

To conclude the proof of Proposition 1 it is enough to prove the following two lemmas.

Lemma 4. *If there exists a correct chain witnessing for X then $\alpha(X) \in \mathrm{L}(\mathcal{A}_1)$.*

Lemma 5. *If $\alpha(X) \in \mathrm{L}(\mathcal{A}_1)$ then there exists a correct chain witnessing for X.*

Proof of Lemma 4 Consider a correct chain $(v_n)_{n\in\omega}$ witnessing for X. Assume that $I \subseteq \omega$ is an infinite set such that for $n \in I$ we have $v_n \in X$ and $v_{n+1} \le_{\text{inf}} v_n\mathtt{L}$. Let us construct inductively a run ρ of $\mathcal{A}_1$ on $\alpha(X)$. The invariant is that for each $n \in \omega$ the configuration of ρ before reading the n-th phase of $\alpha(X)$ is of the form (q_0, c_n) with $c_n \ge e(v_n)$. To define ρ it is enough to decide which block to choose from an n-th phase of $\alpha(X)$:

– if $n \in I$ then choose the block $B^+\big(-e(v_n), +e(v_n\mathtt{L})\big)$,

– otherwise choose the block $B^{-}\big(-e(v_n), +e(v_n\mathtt{R})\big)$.

Notice that by the invariant, it is allowed to choose the respective blocks as $c_n \geq e(v_n)$. Because of (5) and the fact that $(v_n)_{n\in\omega}$ is a correct chain, the invariant is preserved. As the set I is infinite, the constructed run chooses an accepting block infinitely many times and thus is accepting. □

Proof of Lemma 5 Assume that ρ is an accepting run of $\mathcal{A}_1$ over $\alpha(X)$. For $n = 0, 1, \ldots$ let (q_0, c_n) be the configuration in ρ before reading the n-th phase of $\alpha(X)$ and assume that ρ chooses a block of the form $B^{s_n}\big(-e(v_n), +e(v_n d_n)\big)$ in the n-th phase of $\alpha(X)$. Our aim is to show that $(v_n)_{n\in\omega}$ is a correct chain witnessing for X. First notice that by the construction of $\alpha(X)$ we have $|v_n| = n$.

Clearly, as the counter needs to be non-negative, we have $e(v_n) \leq c_n$. Notice that by (6) we obtain inductively for $n = 0, 1, \ldots$ that $c_n < m(n)$. Therefore, we have

$$\begin{aligned} m(n) \cdot b(v_{n+1}) = e(v_{n+1}) \leq c_{n+1} = \\ = c_n - e(v_n) + e(v_n d_n) < m(n) + e(v_n d_n) = \\ = m(n) + m(n) \cdot b(v_n d_n). \end{aligned}$$

By dividing by $m(n)$ we obtain $b(v_{n+1}) < 1 + b(v_n d_n)$, thus $b(v_{n+1}) \leq b(v_n d_n)$ and therefore $v_{n+1} \leq_{\inf} v_n d_n \leq_{\inf} v_n\mathtt{R}$. Moreover, if $s_n = +$ (i.e. the n-th chosen block is accepting) then $v_n \in X$ and $d_n = \mathtt{L}$. Therefore, as ρ chooses infinitely many accepting blocks, $(v_n)_{n\in\omega}$ is witnessing for X. □

This concludes the proof of Proposition 1.

5 Inherent non-determinism

In this seciton we formally state and prove Corollary 1. It is expressed in the same spirit as the corresponding Theorem 5.5 in [11]: we consider an abstract model of automata $\mathcal{A}$ with a countable set of configurations C, an initial configuration $c_{\mathrm{I}} \in C$, a transition relation $\delta \subseteq C \times A \times C$, and an acceptance condition $W \subseteq C^{\omega}$. The notions of a run $\mathrm{run}(\alpha, \rho)$; an accepting run $\mathrm{acc}(\rho)$; and the language $\mathrm{L}(\mathcal{A})$ are defined in the standard way. Thus, under the assumption that the acceptance condition W is Borel, the set

$$P \stackrel{\text{def}}{=} \big\{(\alpha, \rho) \in A^{\omega} \times C^{\omega} \mid \mathrm{run}(\alpha, \rho) \wedge \mathrm{acc}(\rho)\big\},$$

as in (1) is also Borel. The assumptions that the machine is deterministic, unambiguous, or countably-unambiguous imply that the cardinality of the sections $P_{\alpha} \stackrel{\text{def}}{=} \{\rho \mid (\alpha, \rho) \in P\}$ for $\alpha \in A^{\omega}$ is at most countable. Therefore, the following *small section theorem* by Lusin and Novikov applies.

Theorem 5 (see [12, Theorem 18.10]). *Let X, Y be standard Borel spaces and let $P \subseteq X \times Y$ be Borel. If every section P_x is countable, then P has a Borel uniformization and therefore $\pi_X(P)$ is Borel.*

Therefore, we know that $\mathrm{L}(\mathcal{A}) = \pi_{A^\omega}(P)$ is Borel. Thus, no such machine can recognise $\mathrm{L}(\mathcal{A}_1)$ for the Büchi VASS $\mathcal{A}_1$ from Section 4, as that language is non-Borel.

6 Concluding remarks

The core result of this paper is a technique of encoding a $\mathbf{\Sigma}^1_1$-complete set in a *monotone* way using only one partially blind counter — Proposition 1. This shows that even in that restricted case, the non-determinism of the machines is inherent, and cannot be simulated by any restricted form (like countable-unambiguity).

The question whether one counter Büchi VASS recognise languages at all levels of the Wadge hierarchy that are occupied by non-deterministic Büchi Turing machines (see [9]) is left open. The construction provided in [9] involves four counters and at the moment it is not clear whether one can reduce that number.

References

1. Bojańczyk, M.: Weak MSO with the unbounding quantifier. Theory Comput. Syst. **48**(3), 554–576 (2011)
2. Bojańczyk, M., Colcombet, T.: Bounds in ω-regularity. In: LICS. pp. 285–296 (2006)
3. Bojańczyk, M., Toruńczyk, S.: Deterministic automata and extensions of weak MSO. In: FSTTCS. pp. 73–84 (2009)
4. Cabessa, J., Duparc, J., Facchini, A., Murlak, F.: The Wadge hierarchy of max-regular languages. In: FSTTCS. pp. 121–132 (2009)
5. Colcombet, T.: Forms of determinism for automata. In: STACS. pp. 1–23 (2012)
6. Duparc, J., Finkel, O., Ressayre, J.P.: Computer science and the fine structure of borel sets. Theoretical Computer Science **257**(1–2), 85–105 (2001)
7. Duparc, J., Finkel, O., Ressayre, J.P.: The Wadge hierarchy of Petri nets w-languages. In: LFCS. pp. 179–193 (2013)
8. Finkel, O.: Borel ranks and Wadge degrees of context free omega-languages. Mathematical Structures in Computer Science **16**(5), 813–840 (2006)
9. Finkel, O.: Wadge degrees of ω-languages of Petri nets (2018)
10. Finkel, O., Skrzypczak, M.: On the topological complexity of w-languages of non-deterministic Petri nets. Inf. Process. Lett. **114**(5), 229–233 (2014)
11. Hummel, S., Skrzypczak, M.: The topological complexity of MSO+U and related automata models. Fundamenta Informaticae **119**(1), 87–111 (2012)
12. Kechris, A.: Classical descriptive set theory. Springer-Verlag, New York (1995)
13. Thomas, W., Lescow, H.: Logical specifications of infinite computations. In: REX School/Symposium. pp. 583–621 (1993)