Deciding complexity of languages via games

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University of Warsaw

Part 1

Regular languages of things

Things

Things labelled by an alphabet A

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Finite / infinite **words**:

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Finite / infinite **words**: (seen as strings, terms, and logical structures)

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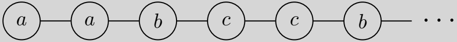

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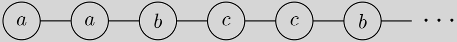

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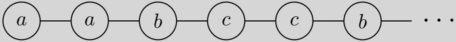

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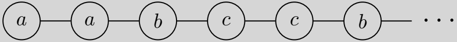

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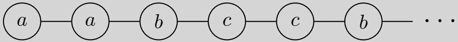

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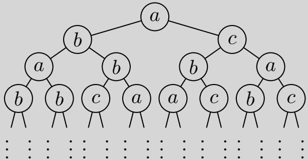

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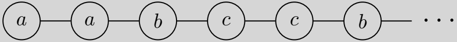

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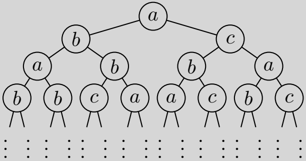

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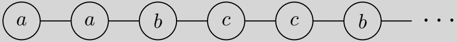

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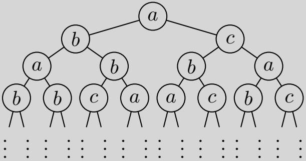

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Monadic Second-order Logic:

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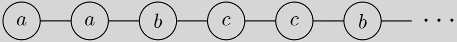

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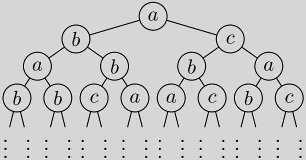

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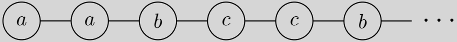

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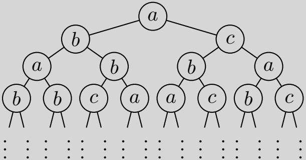

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Regular languages of things

Regular languages of things

Fin. words

Regular languages of things

	logic
Fin. words	MSO

Regular languages of things

logic

Fin. words MSO

$$\mathrm{L}(\varphi) = \{w \mid w \models \varphi\}$$

Regular languages of things

	logic
Fin. words	MSO

Regular languages of things

	logic	**automata**
Fin. words	MSO	DFA

Regular languages of things

	logic	automata
Fin. words	MSO	DFA

$$\mathrm{L}(\mathcal{A}) = \{w \mid \mathcal{A} \text{ accepts } w\}$$

Regular languages of things

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Fin. words	MSO	DFA

Regular languages of things

	logic	automata	algebra
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$$h\colon A^* \to M,\ F \subseteq M$$
$$\mathrm{L}(h) = h^{-1}(F)$$

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Regular languages of things

	logic	automata	algebra	expressions
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Inf. words					

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Regular languages of things

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$\rightsquigarrow$ MSO is universal

$\rightsquigarrow$ complicated structures require complicated devices

$\rightsquigarrow$ infinite trees inherently require non-determinism

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Part 2

Effective characterisations by patterns

Given $L \subseteq A^*$, is L definable in FO?

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$$A^* = {\sim}\varnothing$$
$$(ab)^* = {\sim}\big[\; bA^* \cup A^*a \cup A^*aaA^* \cup A^*bbA^* \big]$$

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$$\Big[\text{If } \mathrm{QD}(\varphi) = k \text{ then } a^{2^k} \models \varphi \text{ iff } a^{2^k+1} \models \varphi\Big]$$

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"pattern method for rigid representations"

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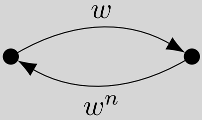

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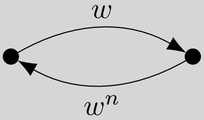

1. Let $L = \mathrm{L}(\mathcal{A})$ for a counter-free $\mathcal{A}$

L is definable in FO

iff

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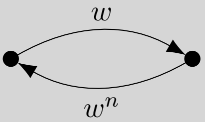


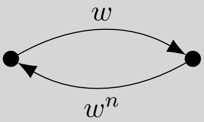
1. Let $L = \mathrm{L}(\mathcal{A})$ for a counter-free $\mathcal{A}$

$\rightsquigarrow$ write φ in FO such that $L = \mathrm{L}(\varphi)$

L is definable in FO

iff

the minimal automaton $\mathcal{A}_L$ for L is counter-free



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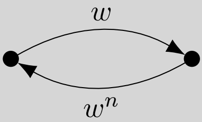

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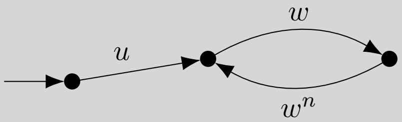

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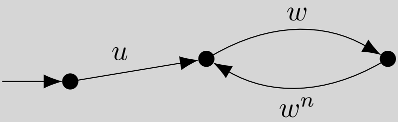

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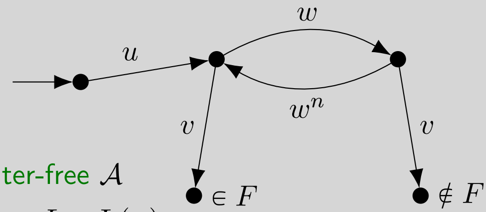

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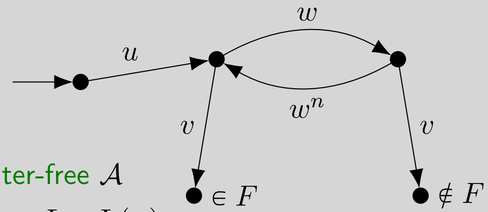

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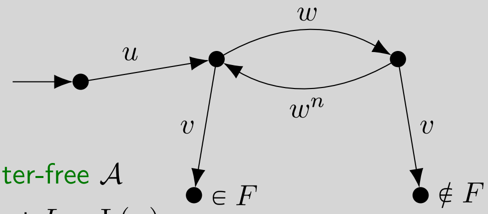

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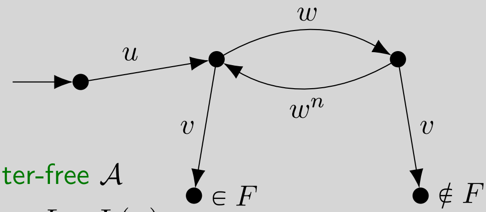

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Examples:

Benedikt, Blumensath, Bojańczyk, Colcombet, Facchini, Idziaszek, Murlak, Niwiński, Pin, Place, Schutzenberger, Segoufin, Straubing, Thérien, Thomas, Walukiewicz, Wilke, Zeitoun, ...

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(Birkhoff)

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- rigid representations needed (Birkhoff)

Rigid representations

Rigid representations

	logic	automata	algebra	expressions	···
Fin. words	MSO	DFA	monoids	regexp	
Inf. words	MSO	det. parity	Wilke alg.	ω-regexp	
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Not mentioned: thin trees, Boolean combinations of open sets, ...

Deterministic parity automata

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Part 3

Games on graphs

Games of: infinite duration, perfect information, finite arena

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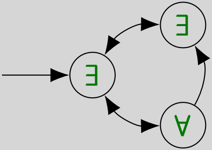

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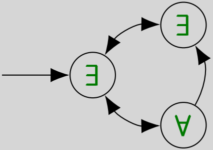

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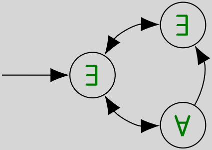

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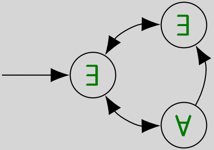

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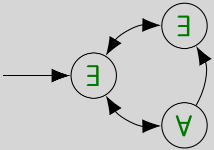

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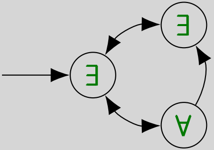

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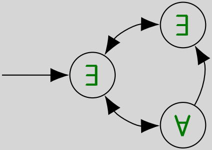

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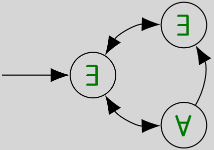

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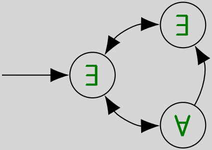

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Also: a winning strategy may require unlimitted memory!

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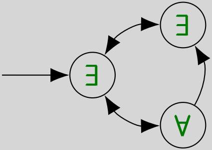

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$\exists$ wins π iff $\pi \in W$ $\forall$ wins π iff $\pi \notin W$

$\rightsquigarrow$ each play has a winner

But: for certain $W \subseteq V^\omega$ none of the players has a winning strategy!

E.g. for infinite XOR, see (Kopczyński, Niwiński [2014])

Also: a winning strategy may require unlimitted memory!

And: it may not be decidable which of the players wins. . .

Theorem (Büchi, Landweber [1969])

If $W \subseteq V^\omega$ is regular then:

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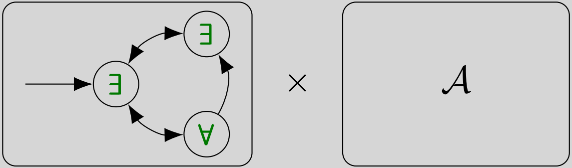

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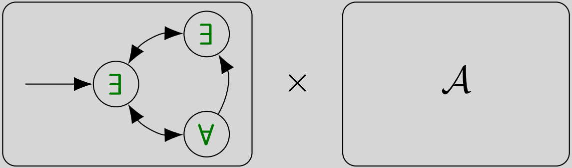

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Parity games are effectively, positionally determined.

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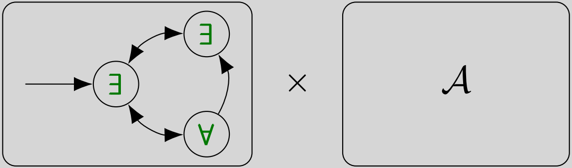

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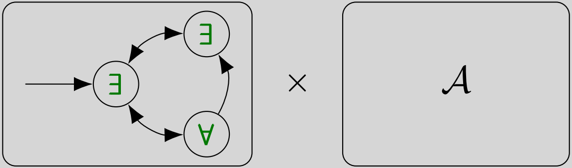

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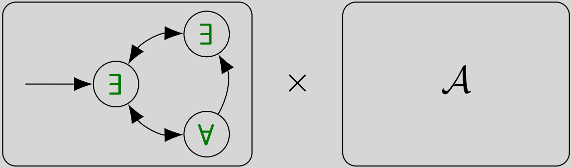

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(Chatterjee, Henzinger, Kupferman, Piterman, Vardi, ...)

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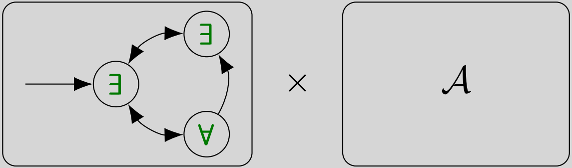

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(Kopczyński [2006]; Zimmermann [2016]; Colcombet, Göller [2016])

Part 4

First examples

Task:

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Input: Regular $L \subseteq A^\omega$ and $i < j$

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Game:

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$\forall$:

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Game:

$\forall$: a_0

$\exists$:

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Game:

$\forall$: $a_0 \in A$

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$\forall$: $a_0 \in A$ $\quad a_1$

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\end{array}
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$$W \stackrel{\text{def}}{=} \Big\{a_0 p_0 a_1 p_1 \cdots \mid \big(a_0 a_1 \cdots \in L\big) \iff \big(\limsup_{n\to\infty} p_n \equiv 0 \pmod 2\big)\Big\}$$

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$\rightsquigarrow$ a witness of hardness for L

Task: **(Wadge+Wagner hierarchy)**

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Input: Regular $L \subseteq A^*$ and k

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Output: Can L be defined by a regexp of star-height $\leqslant k$ (no complementation here!)

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- if $\exists$ wins $\mathcal{G}$ then her memory gives limitedness

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- construct a game $\mathcal{G}$ with regular W
- if $\exists$ wins $\mathcal{G}$ then her memory gives limitedness
- if $\forall$ wins $\mathcal{G}$ then there is **no** limitedness

Part 5

More examples (infinite trees)

Rigid representations

	logic	**automata**	**algebra**	**expressions**	...
Fin. words	MSO	DFA	monoids	regexp	
Inf. words	MSO	det. parity	Wilke alg.	ω-regexp	
Fin. trees	MSO	det. bottom-up	forest alg.	tree regexp	
Det. lang of inf. trees	—	det. top-down	—	—	
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⇝ infinite trees inherently require non-determinism

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⇝ infinite trees inherently require non-determinism

(Niwiński, Walukiewicz [1996]; Carayol, Löding [2010])

(Bilkowski, S. [2013]; Blumensath [2013])

Task (Rabin-Mostowski index problem):

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Input: Regular language of inf. trees L and $i < j$

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Theorem (Colcombet, Löding [2008])

Reduction to domination of cost functions

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Theorem (Colcombet, Löding [2008])

Reduction to domination of cost functions

[not known to be decidable]

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Reduction to domination of cost functions

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[framework of domination games]

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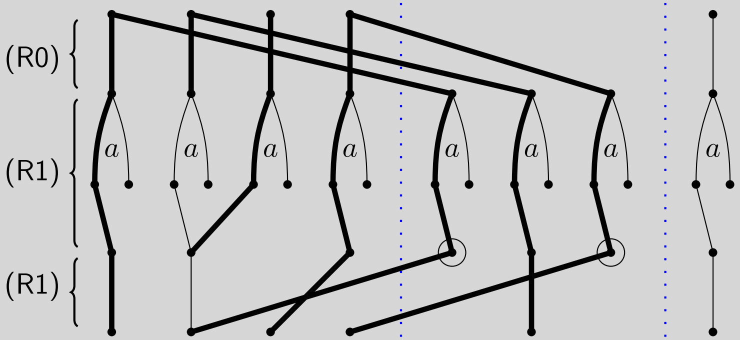

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$\exists$ wins $\Rightarrow L$ is $(1,2)$-definable and **Borel** $\quad$ $\forall$ wins $\Rightarrow L$ is **not** $(1,2)$-definable and **non-Borel**

Part 6

Last example(s)

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Proof

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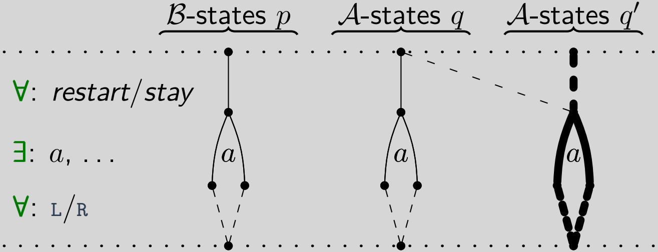

Gameplay. . .

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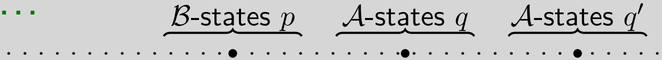
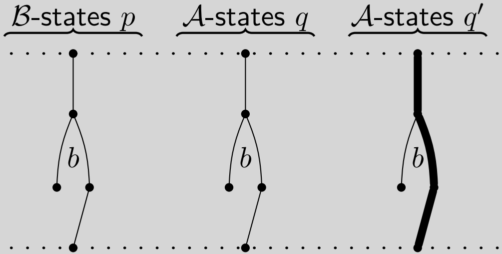


Gameplay...

Round 0

- $\forall$: *stay*
- $\exists$: b, ...
- $\forall$: R



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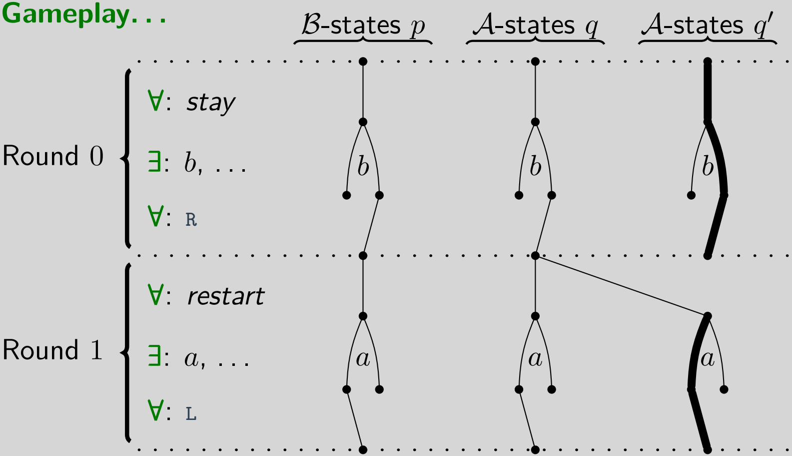

$\mathcal{B}$-states p
$\mathcal{A}$-states q
$\mathcal{A}$-states q'
Round 0
∀: *stay*
∃: b, ...
∀: R
Round 1
∀: *restart*
∃: a, ...
∀: L
b
b
b
a
a
a

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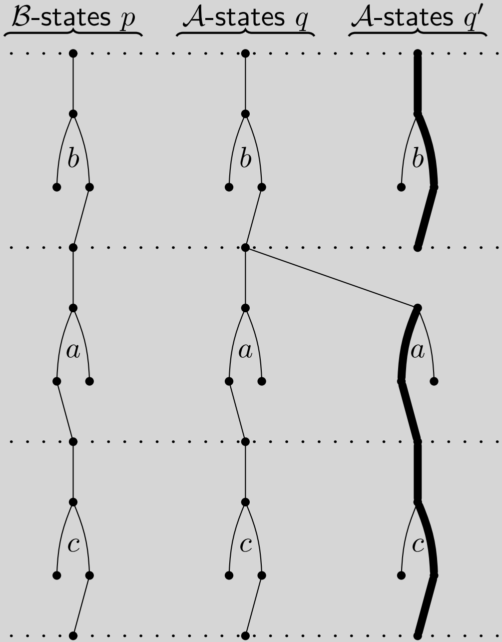
- $\forall$: L

Round 2
- $\forall$: *stay*
- $\exists$: c, ...
- $\forall$: R



Winning condition

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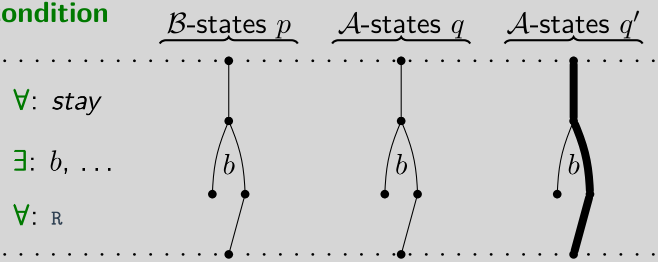

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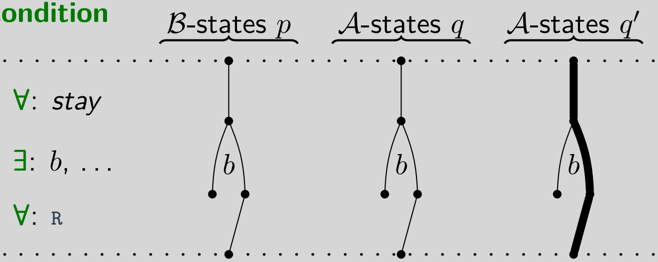
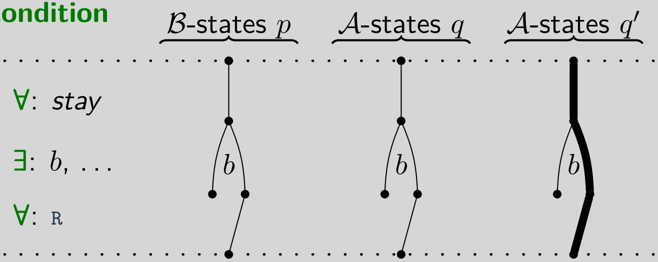


(WR) ∀ restarted infinitely many times

Winning condition



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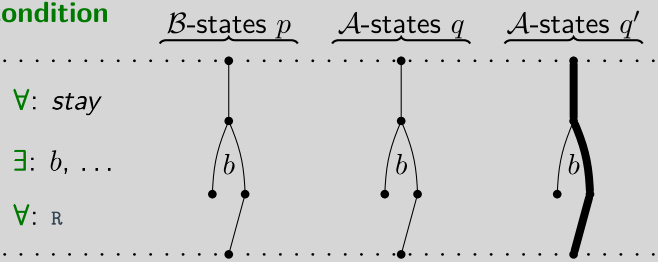

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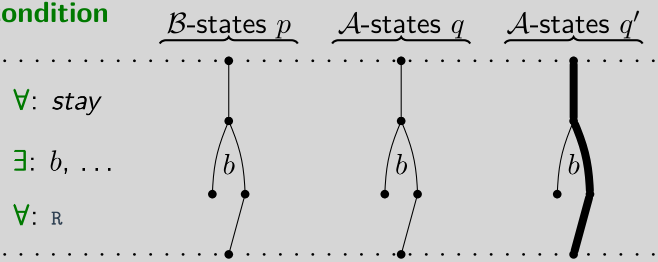

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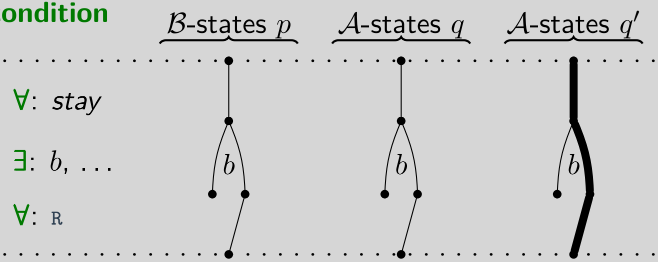

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$\rightsquigarrow$ regular condition over infinite words

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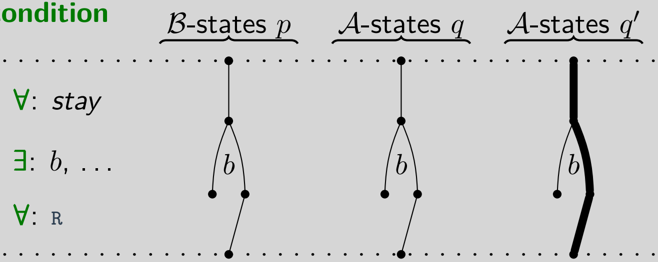

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⇝ regular condition over infinite words

⇝ we can solve $\mathcal{F}$

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Proof

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$\rightsquigarrow$ a weak alternating $(0,2)$ automaton for L

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Confront it with a family of quasi-strategies of $\forall$

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} A complete proof

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A complete proof **not** using properties on which the game $\mathcal{F}$ is based

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[dealternation]

Another example

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Making the Banach-Mazur game regular

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Open problem: what about general regular tree languages?

Summary

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→ characterising which languages are simple

Summary

$\longrightarrow$ characterising which languages are simple

$\longrightarrow$ pattern method (rigid representatons: det. aut. / algebra)

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pattern found

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strategy of ∃

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Summary

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→ pattern method (rigid representatons: det. aut. / algebra)

- pattern found
  ⇝ L is hard
- pattern missing
  ⇝ L is simple

→ games (may deal with non-determinism)

- strategy of ∃
  ⇝ L is hard
- strategy of ∀
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↙ ↘

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strategy of $\exists$ $\rightsquigarrow$ L is hard

strategy of $\forall$ $\rightsquigarrow$ L is simple

→ **no** general recipe for design

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Conjecture: Every class of languages has a game characterisation