Connecting decidability and complexity for MSO logic

Michał Skrzypczak

University of Warsaw

Part 0

MSO logic

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$$a < b \leftrightarrow (\bigvee A)[b \in A \wedge (\bigwedge x)(x \in A \to x' \in A) \wedge a \notin A]$$

and

$$a \equiv 0 \pmod 2 \leftrightarrow (\bigwedge A)[0 \in A \wedge (\bigwedge x)(x \in A \to x'' \in A) \to a \in A].$$

Specifically, Tarski has proposed the following two problems.

PROBLEM 1. Is it possible to give a restricted set-theoretical definition of addition of natural numbers in terms of successor?

PROBLEM 2. Is there a decision method for the arithmetic of natural numbers based on the notion of successor and using restricted set theory?

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Boris A. Trakhtenbrot [1962]

"Finite automata and the logic of one-place predicates"

Siberian Math. J., 3:103–131, 1962.

(English translation in: AMS Transl. 59 (1966) 23–55.)

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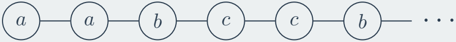
$\alpha =$ 

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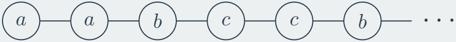
$\alpha =$ 

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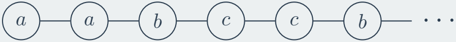
$\alpha =$ 

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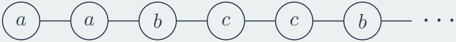
$\alpha =$ 

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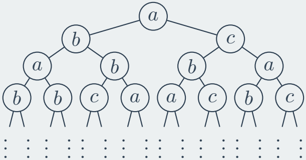
$t =$ 

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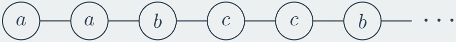
$\alpha =$ 

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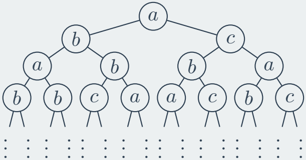
$t =$ 

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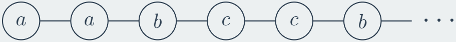
$\alpha =$ 

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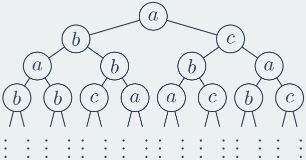
$t =$ 

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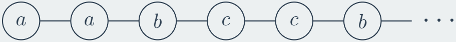
$\alpha =$ 

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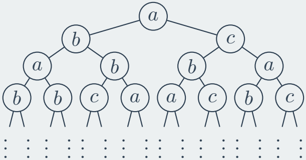
$t =$ 

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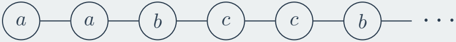
$\alpha =$ 

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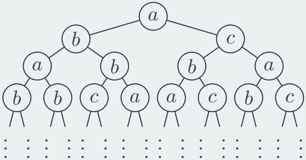
$t =$ 

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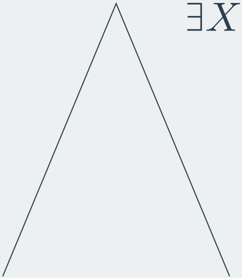

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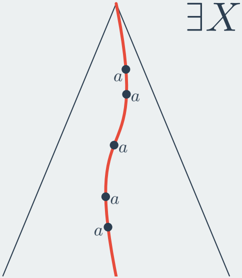

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- Construct automata recognising basic sets:

$$\mathrm{L}(\mathcal{A}_{\in}) = \big\{\{x\} \otimes Y \text{ over } \{0,1\}^2 \mid x \in Y\big\}$$

- Implement logical operations in automata:

  Connectives: $\mathcal{A}$, $\mathcal{B}$ over $A \rightsquigarrow \neg\mathcal{A}$ and $\mathcal{A} \vee \mathcal{B}$ over A

Decidability

Theorem (Büchi [1962] / Rabin [1969])

The MSO theory of (ω, s) / $(\{0,1\}^{<\omega}, s_0, s_1)$ is decidable.

Proofs: Automata $\mathcal{A}$

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- Transform φ into $\mathcal{A}$ and check if $\mathrm{L}(\mathcal{A}) = \emptyset$

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the execution $\mathcal{P}(\varphi)$ terminates

and returns TRUE iff φ is true

Part 1

Topological complexity

Topology of infinite words / trees

Topology of infinite words / trees

$\left(2 \leqslant |A| < \infty\right)$

Topology of infinite words / trees

$\big(2 \leqslant |A| < \infty\big)$

words — A^ω

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words — A^{ω} $\qquad$ $A^{(\{\mathtt{L},\mathtt{R}\}^{<\omega})}$ — trees

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$$\{0,1\}^\omega$$

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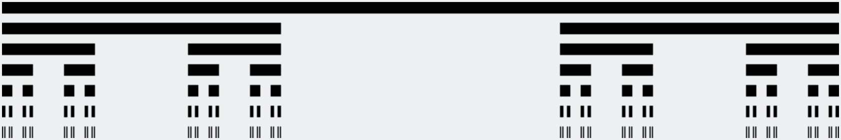

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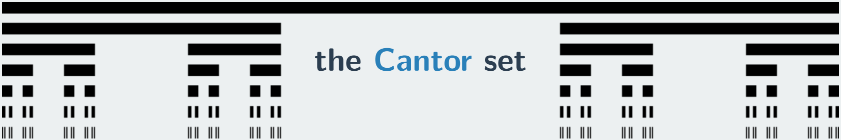

Topology of infinite words / trees

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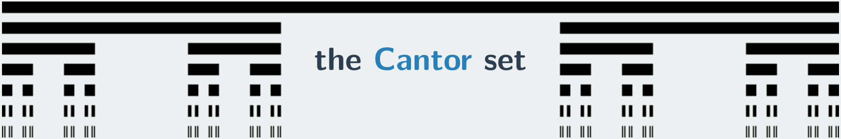
words — A^{ω} $\cong$ $\{0,1\}^{\omega}$ $\cong$ $A^{(\{\mathtt{L},\mathtt{R}\}^{<\omega})}$ — trees

$$\{0,1\}^{\omega}$$

$\cong$



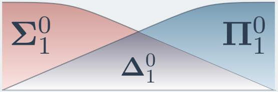
$\mathrm{L}(\varphi) \cong$ set of points

Descriptive set theory

Descriptive set theory

Start from simple sets

$\rightsquigarrow$ open ($\mathbf{\Sigma}^0_1$) and closed ($\mathbf{\Pi}^0_1$)



Descriptive set theory

Start from simple sets

$\rightsquigarrow$ open ($\mathbf{\Sigma}^0_1$) and closed ($\mathbf{\Pi}^0_1$)

Apply countable unions ($\bigcup$)

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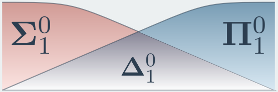

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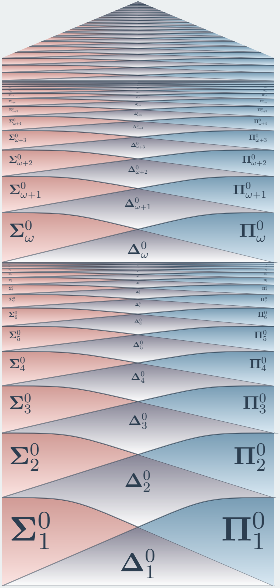

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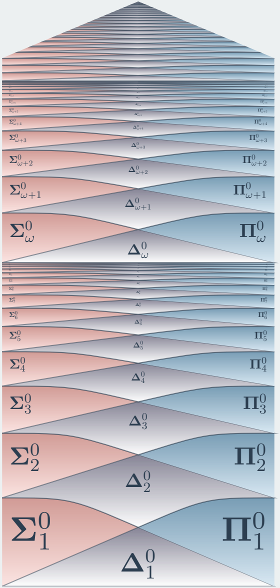
⇝ Borel sets: $\mathbf{\Sigma}^0_\eta$, $\mathbf{\Pi}^0_\eta$ for $\eta < \omega_1$



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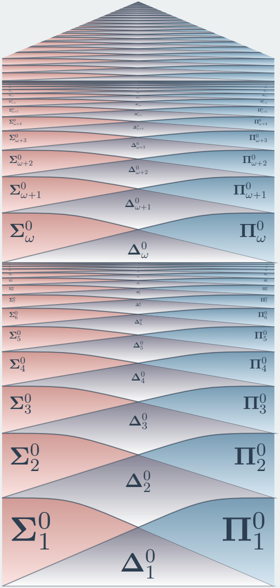
Apply projection and co-projection



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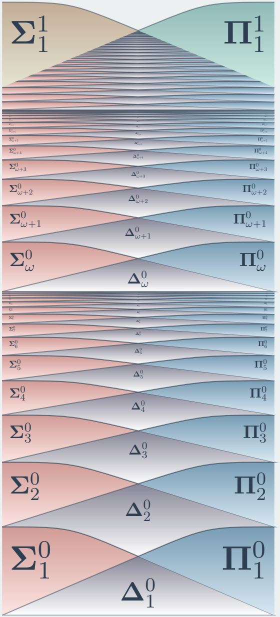

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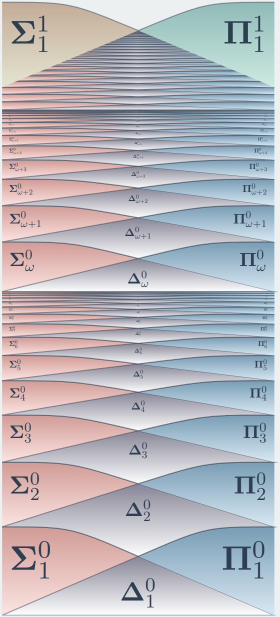
By induction



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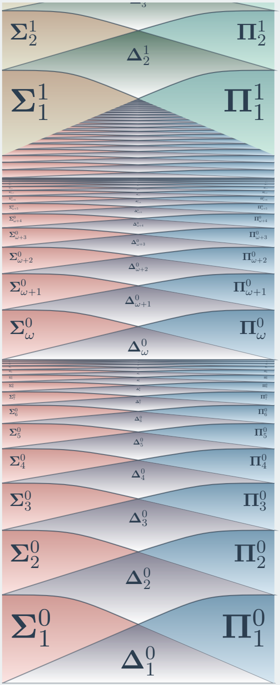
Apply projection and co-projection

$\rightsquigarrow$ analytic ($\mathbf{\Sigma}^1_1$) and co-analytic ($\mathbf{\Pi}^1_1$)

By induction

$\rightsquigarrow$ projective sets: $\mathbf{\Sigma}^1_n$, $\mathbf{\Pi}^1_n$ for $n < \omega$



Upper bounds

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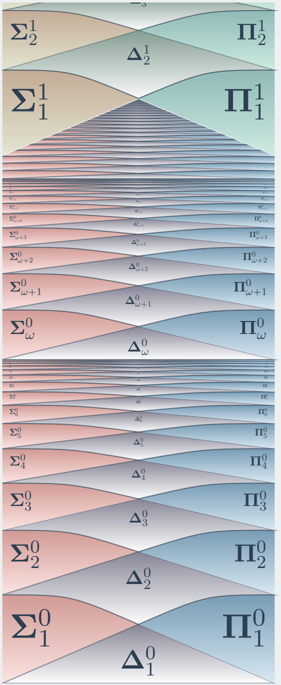
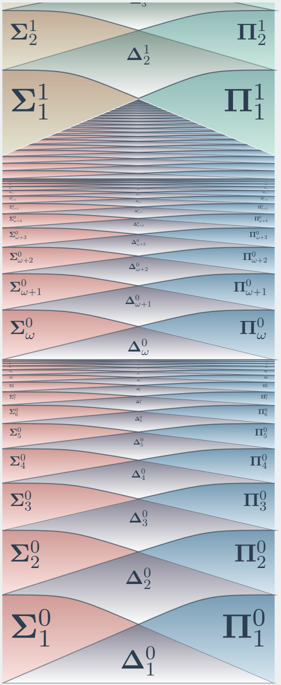


Upper bounds

$\exists X \rightsquigarrow$ projection $A \times \{0,1\} \to A$



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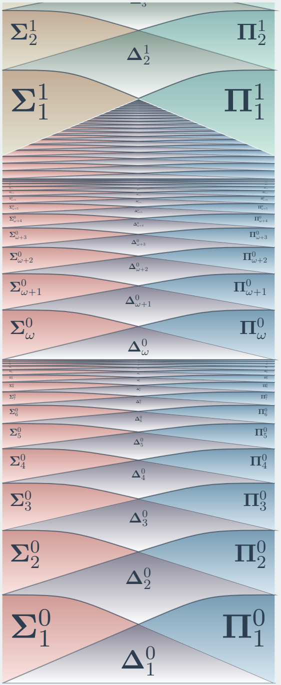

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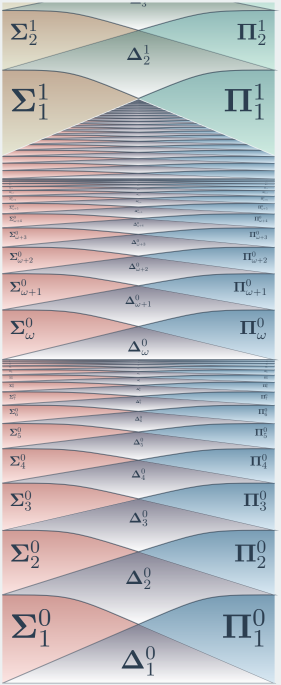

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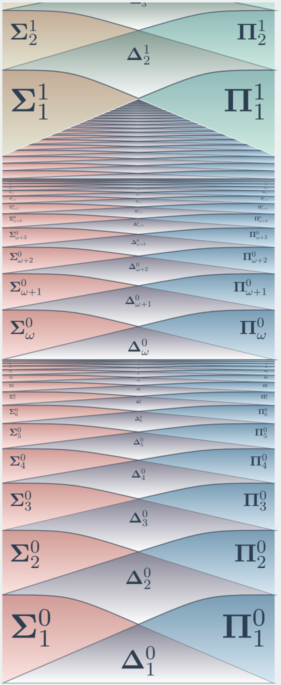

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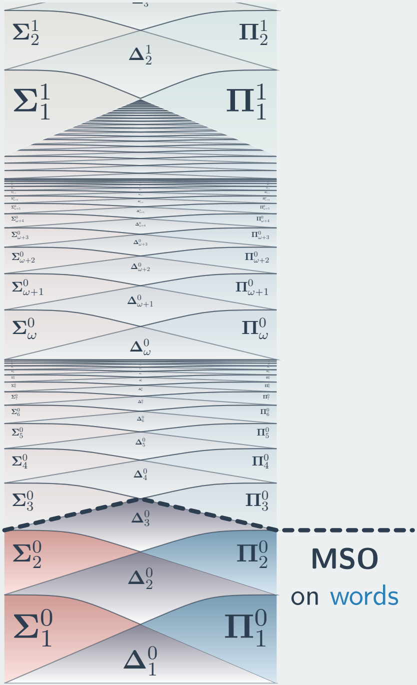

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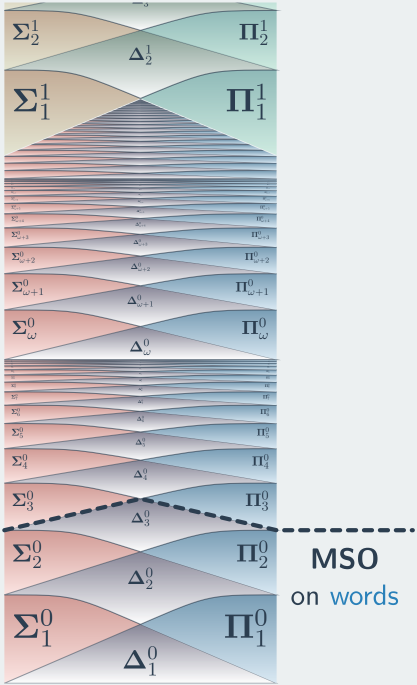

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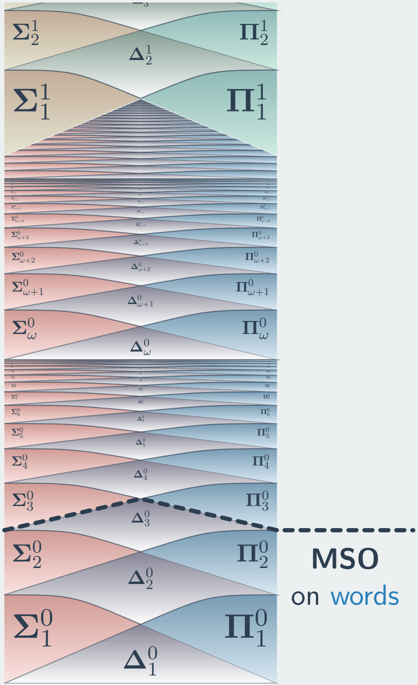

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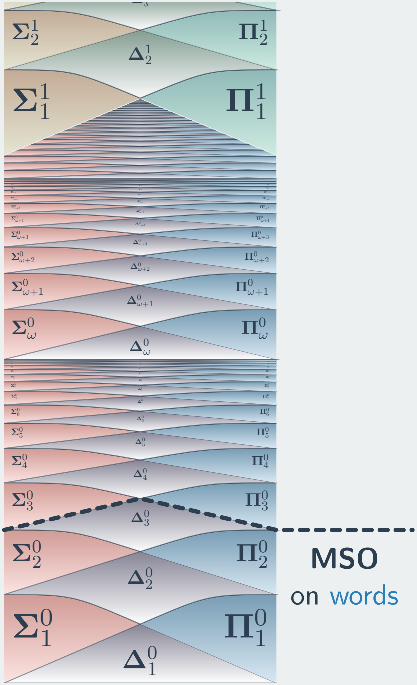

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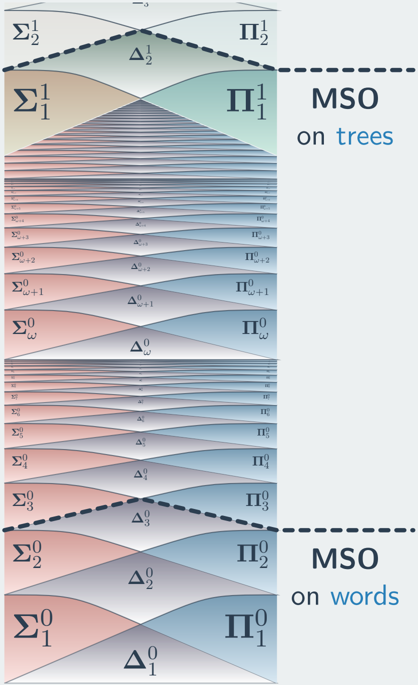

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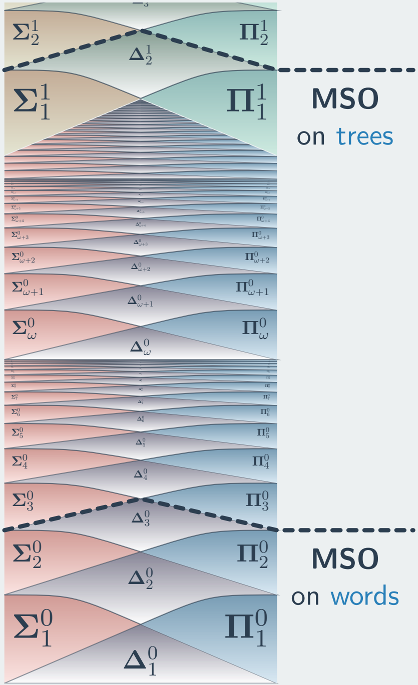

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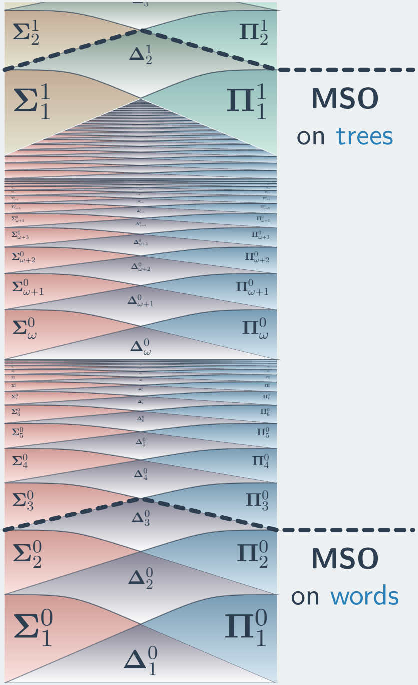
Theorem (Niwiński [1985])

There exists a non-Borel ($\mathbf{\Sigma}^1_1$-compl.) set definable in MSO.



Topological properties

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Take φ — MSO formula on trees

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Topological properties

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Is $\mathrm{L}(\varphi)$ measurable?

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Theorem (Gogacz, Michalewski, Mio, S. [2014])

If $\varphi \in$ MSO on trees then $\mathrm{L}(\varphi)$ is an **$\mathcal{R}$-set**.

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Is the Gale-Stewart game on $\mathrm{L}(\varphi)$ determined?

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Extensions of MSO

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"$\varphi(X)$ holds for arbitrarily big finite sets"

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Large expressive power: cost functions, distance automata, ...

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Example

The delays between REQUEST and RESPONSE are uniformly bounded.

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Example

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Conjecture (Bojańczyk [2004])

The MSO+U theory of (ω, s) is decidable.

Extensions of MSO

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$\rightsquigarrow$ no reasonable automaton model for MSO+U

Part 1'

Topological complexity vs. decidability

Theorem (Shelah [1975]; Gurevich, Shelah [1982])

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Corollary (Bojańczyk, Gogacz, Michalewski, S. [2014])

It is consistent with ZFC that

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Theorem (Bojańczyk, Parys, Toruńczyk [2016])

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Part 2

Reverse mathematics

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- define depth of a formula: alternations of $\exists/\vee$ and $\forall/\wedge$
- formulate decidability of depth-n fragments for specific n
- study these sentences

Part 2.a

Reversing Büchi

(Kołodziejczyk, Michalewski, Pradic, S. [2016])

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[Ramsey's Theorem for Pairs]

$\mathrm{RT}^2_{<\infty}$

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[Ramsey's Theorem for Pairs]

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[Weak König's Lemma]

WKL_0

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[Ramsey's Theorem for Pairs]

$\mathrm{RT}^2_{<\infty}$

$\Downarrow$

[Additive Ramsey's Theorem]

$\mathrm{ART}^2_{<\infty}$

[Weak König's Lemma]

WKL_0

DEC_5

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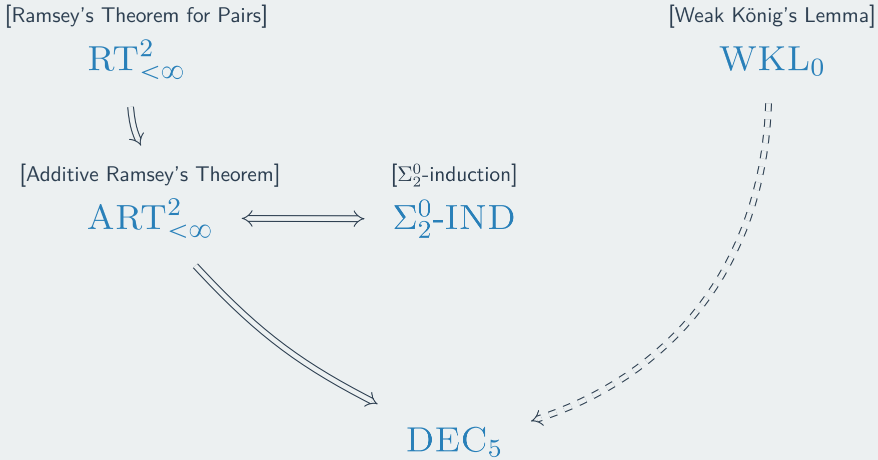

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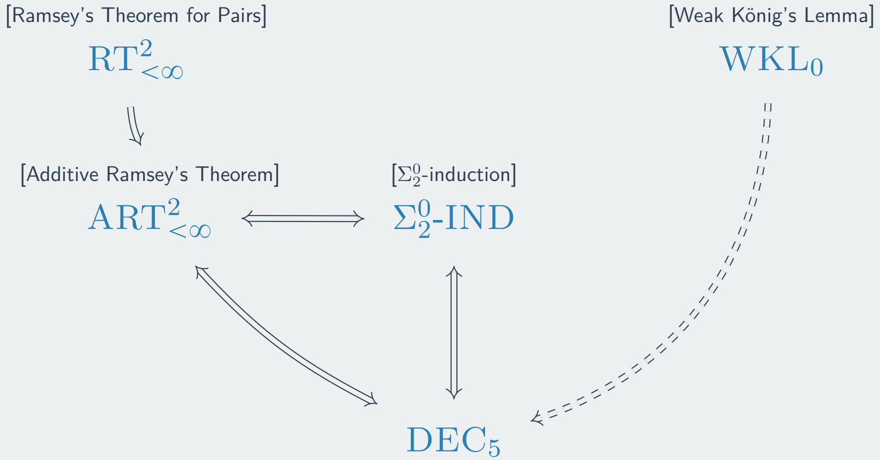

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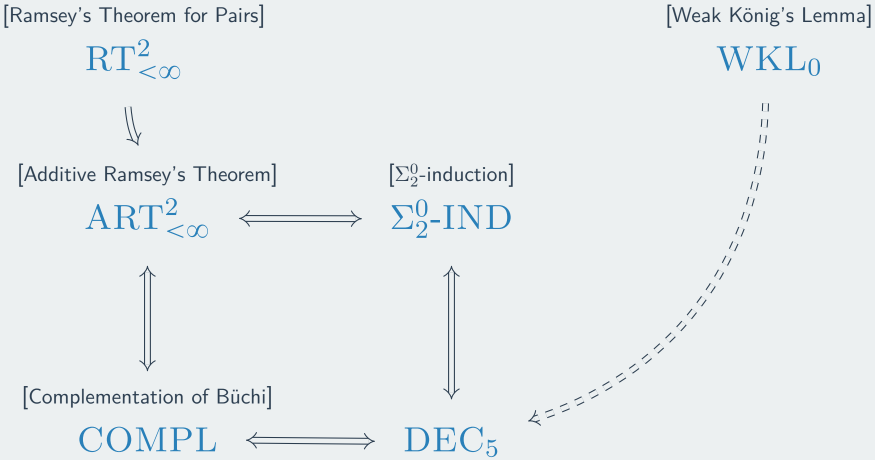

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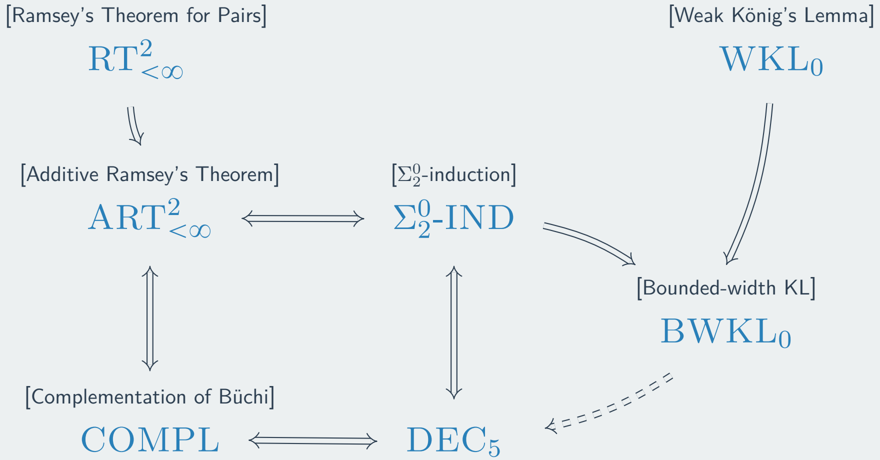

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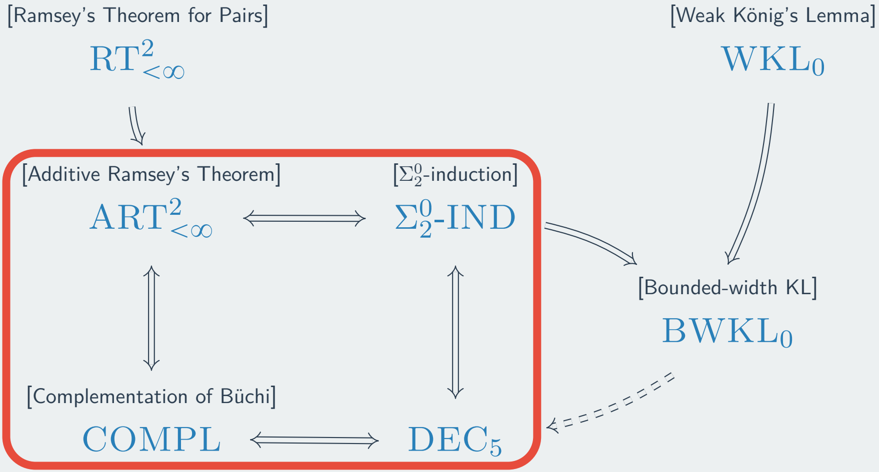

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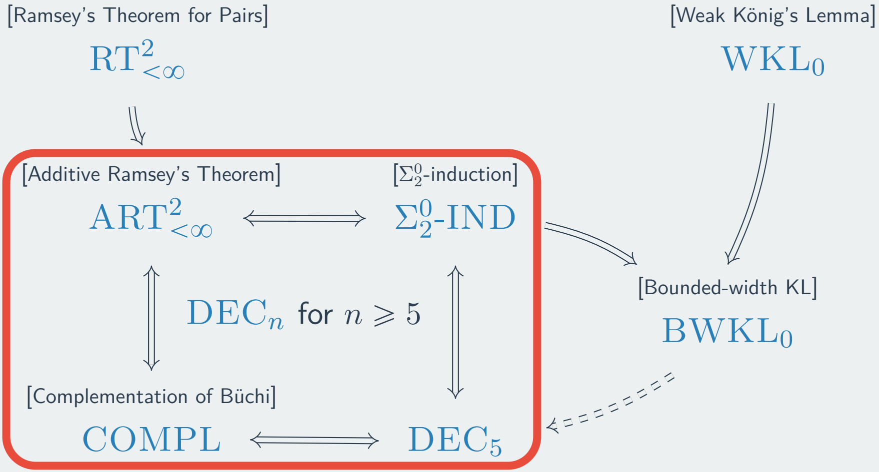

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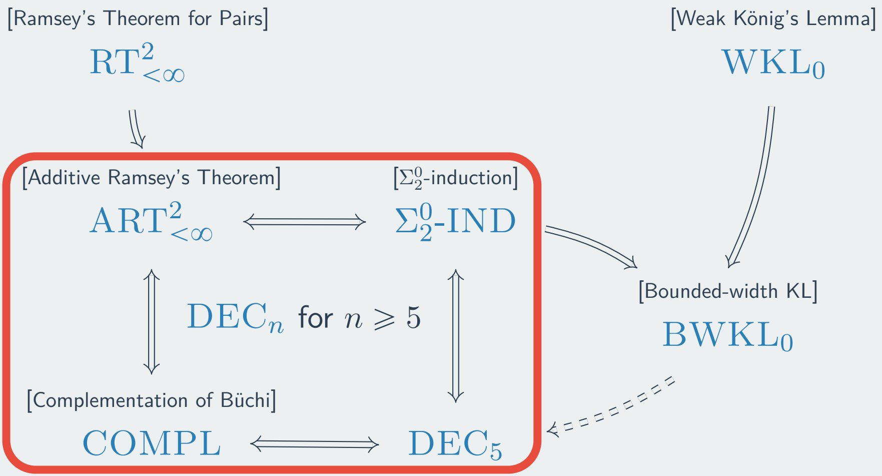

Theorem (Kołodziejczyk, Michalewski, Pradic, S. [2016])

The following holds over RCA_0:

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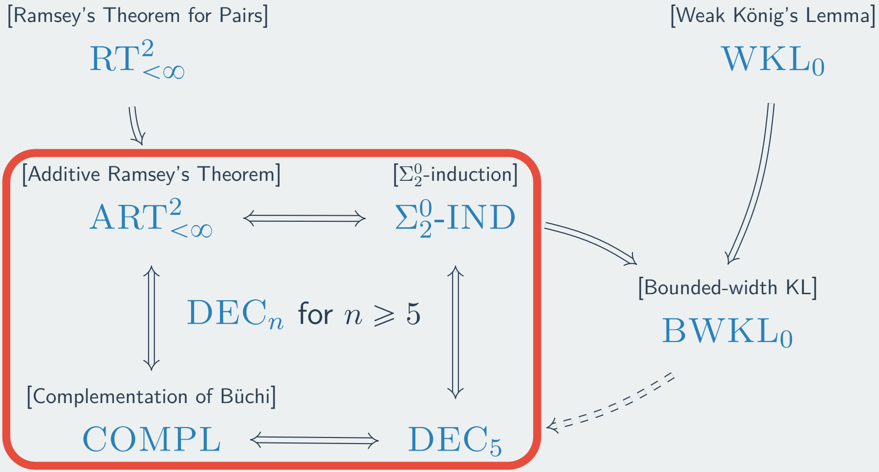

Part 2.b

Reversing Rabin

(Kołodziejczyk, Michalewski [2016])

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  then $\mathrm{RCA}_0 \vdash \big(\mathrm{COMPL} \iff \boldsymbol{\Psi}\big)$

(where COMPL is Rabin's complementation)

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