Connecting decidability and complexity for MSO logic

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NATIONAL SCIENCE CENTRE POLAND

Structures

Structures over an alphabet A

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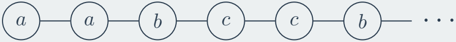
$\alpha =$ 

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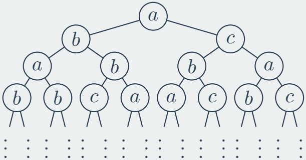
$t =$ 

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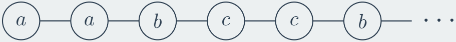
$\alpha =$ 

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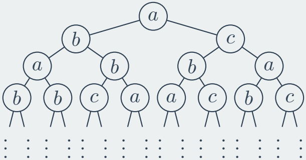
$t =$ 

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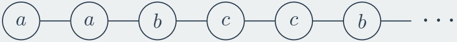
$\alpha =$ 

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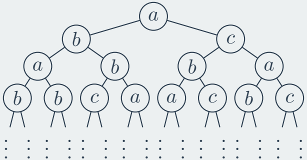
$t =$ 

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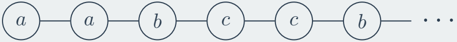
$\alpha =$ 

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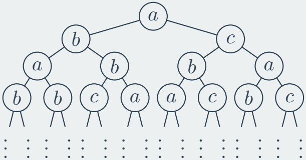
$t =$ 

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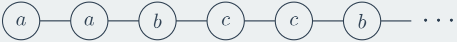
$\alpha =$ 

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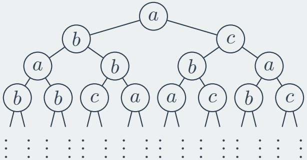
$t =$ 

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Part 0

MSO logic

History: why MSO logic?

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Alfred Tarski has proposed (in lectures) consideration of an intermediate type of definition, in which sets of natural numbers but no other sets are allowed. Thus we will have variables $a, b, c, \cdots$ which represent natural numbers, and variables $A, B, C, \cdots$ which represent sets of natural numbers. The term restricted set theory will refer to the use of just these types of variables. A definition using such variables will be called a *restricted set-theoretical definition*. As examples of definitions of this type, we may give

$$a < b \leftrightarrow (\bigvee A)[b \in A \wedge (\bigwedge x)(x \in A \rightarrow x' \in A) \wedge a \notin A]$$

and

$$a \equiv 0 \pmod 2 \leftrightarrow (\bigwedge A)[0 \in A \wedge (\bigwedge x)(x \in A \rightarrow x'' \in A) \rightarrow a \in A].$$

Specifically, Tarski has proposed the following two problems.

PROBLEM 1. Is it possible to give a restricted set-theoretical definition of addition of natural numbers in terms of successor?

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φ defines a language (set of words / trees):

$$\mathrm{L}(\varphi) \stackrel{\text{def}}{=} \{M \mid M \models \varphi\}$$

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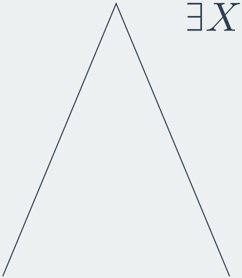

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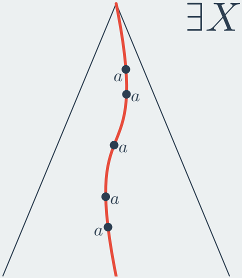

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— input a formula φ on words / trees ($_{\text{with no letter predicates}}$)

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$$\exists X_a \ldots X_z. \left(X\text{'s are a partition}\right) \wedge \varphi[a(x) \rightarrow x \in X_a, \ldots]$$

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$$\Big(\textbf{union}, \textbf{complementation } (\,!\,), \text{ and } \textbf{projection } (\,!\,)\Big)$$

- Transform φ into $\mathcal{A}$ and check if $\mathrm{L}(\mathcal{A}) = \varnothing$ ∎

$\Longrightarrow$ [There exists an algorithm $\mathcal{P}$ such that for every formula φ the execution $\mathcal{P}(\varphi)$ terminates]

Decidability

Theorem (Büchi [1962] / Rabin [1969])

The MSO theory of (ω, s) / $(\{\mathtt{L}, \mathtt{R}\}^*, s_{\mathtt{L}}, s_{\mathtt{R}})$ is decidable.

Proofs: automata $\mathcal{A}$

- Construct automata for basic languages:

$$\mathrm{L}(\mathcal{A}_{\in}) = \big\{\{x\} \otimes Y \text{ over } \{0,1\}^2 \mid x \in Y\big\}$$

- Implement logical operations in automata

  (**union**, **complementation** (**!**), and **projection** (**!**))

- Transform φ into $\mathcal{A}$ and check if $\mathrm{L}(\mathcal{A}) = \varnothing$ ∎

$\Longrightarrow$ [There exists an algorithm $\mathcal{P}$
such that for every formula φ
the execution $\mathcal{P}(\varphi)$ terminates
and returns TRUE **iff** φ is true.]

Part 1

Topological complexity

Topology of infinite words / trees

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$\left(2 \leqslant |A| < \infty\right)$

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words — A^ω

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words — A^{ω} $\qquad$ $A^{(\{\mathtt{L},\mathtt{R}\}^*)}$ — trees

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$$A^\omega \cong \{0,1\}^\omega \cong A^{(\{\mathtt{L},\mathtt{R}\}^*)}$$

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$\big(2 \leqslant |A| < \infty\big)$

words — A^ω $\qquad$ $A^{(\{\mathtt{L},\mathtt{R}\}^*)}$ — trees

$\cong \qquad \cong$

$$\{0,1\}^\omega$$

$\cong$

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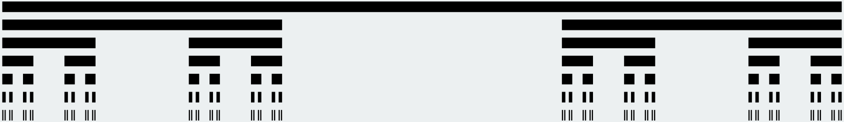
$\cong$



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$\cong$ $\quad$ $\cong$

$$\{0,1\}^{\omega}$$

$\cong$

$\mathrm{L}(\varphi) \cong$ set of points

Descriptive set theory

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Start from simple sets

⇝ $L \in \boldsymbol{\Delta}^0_1$ **iff** L depends on finite prefix

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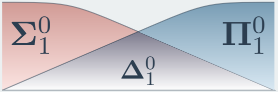

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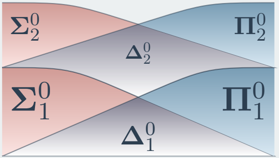

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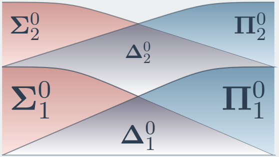

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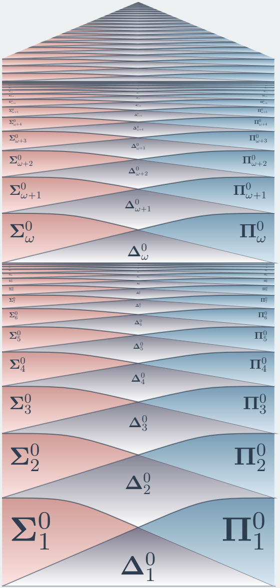

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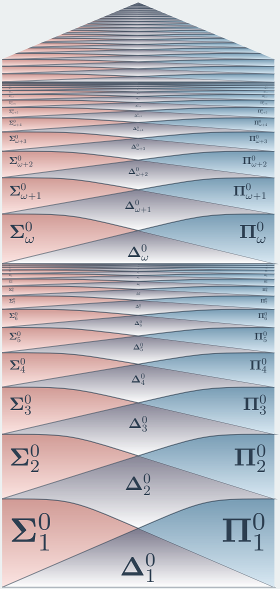

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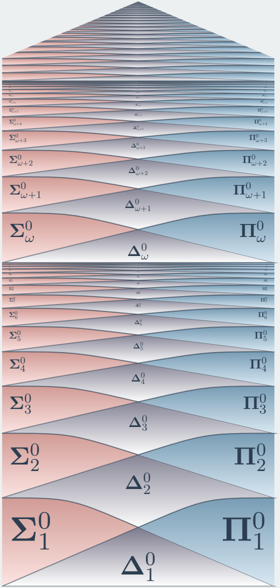

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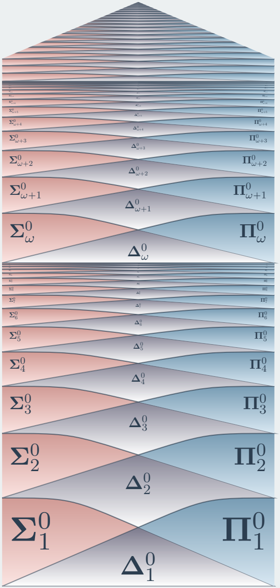

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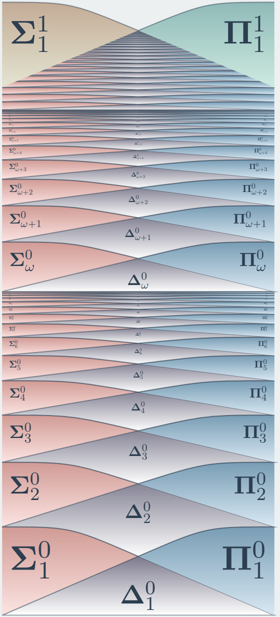

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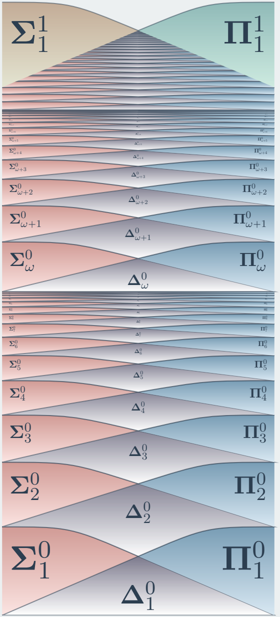

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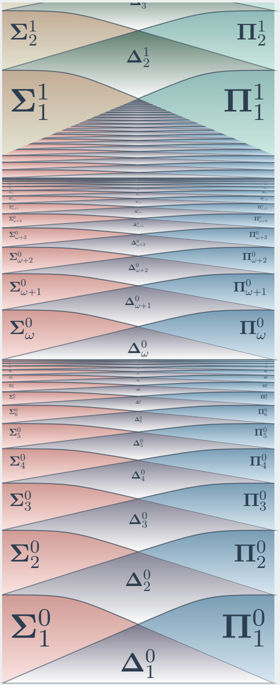

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Apply projection and co-projection

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By induction

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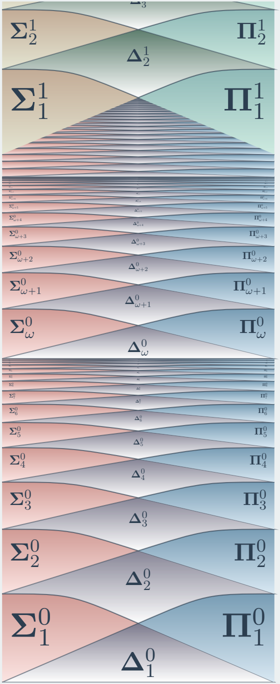

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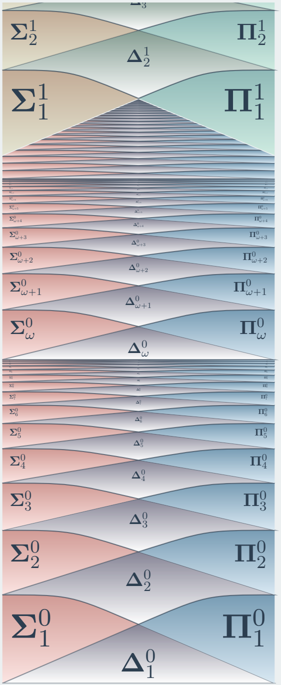
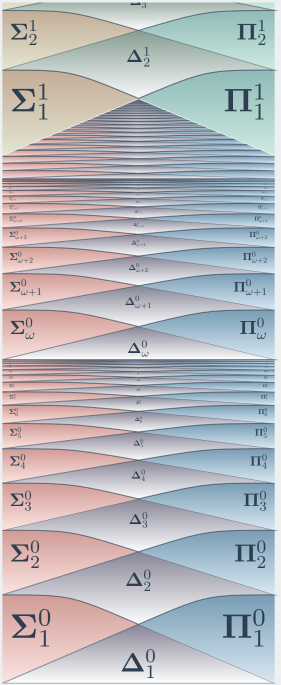


Upper bounds

$\exists X \rightsquigarrow$ projection $A \times \{0,1\} \to A$



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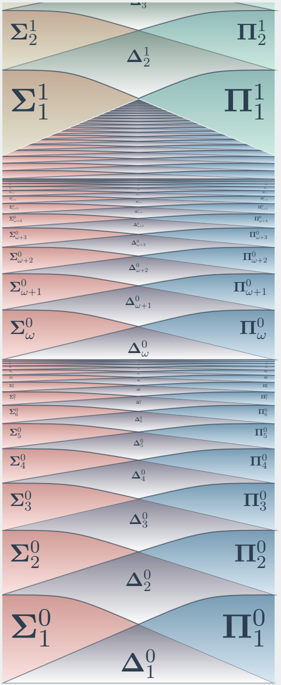

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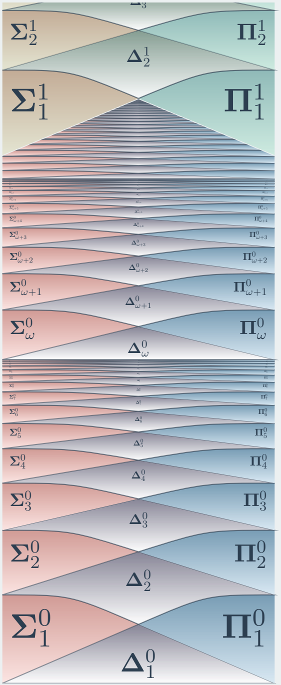

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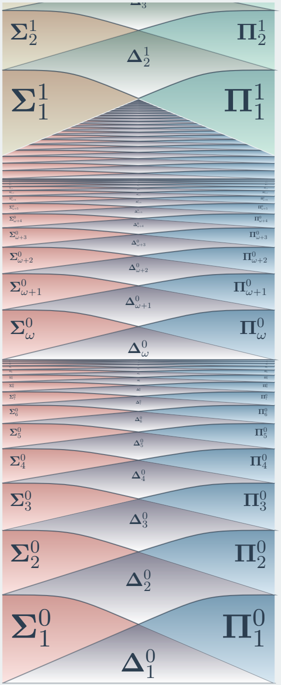
MSO on words $\equiv$ deterministic aut. $\mathcal{A}$



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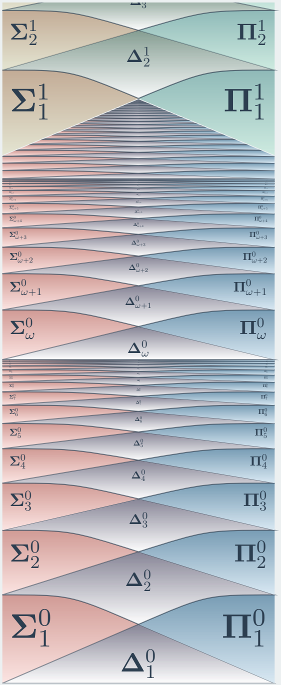

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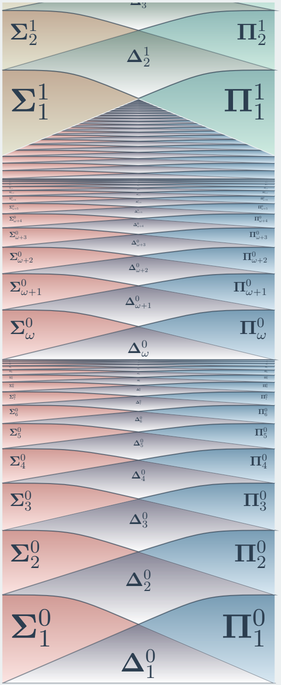

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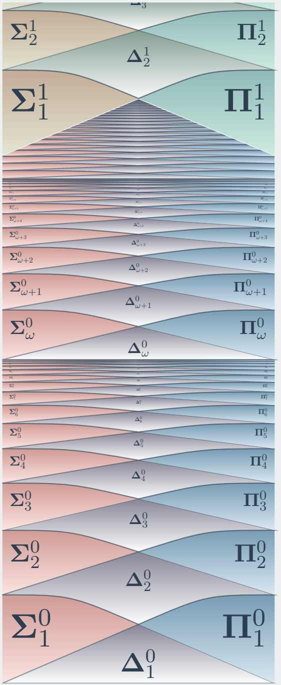

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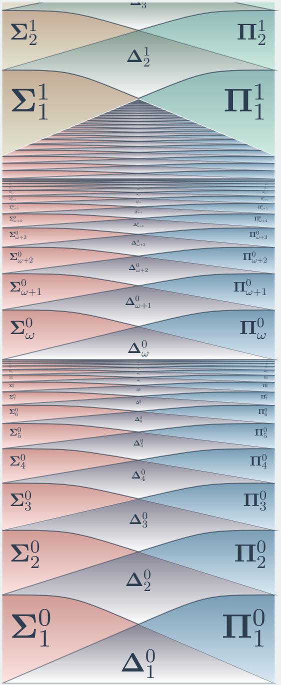

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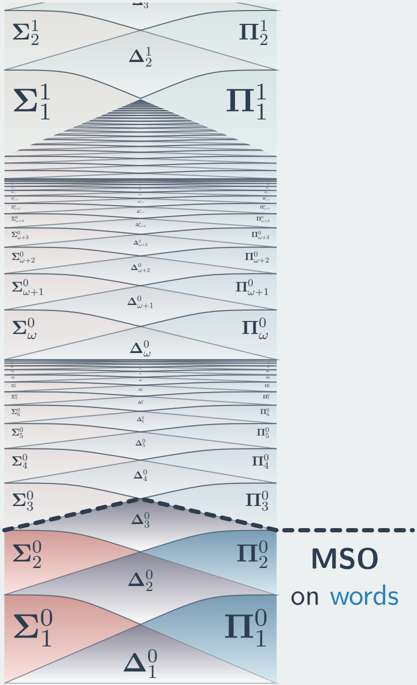

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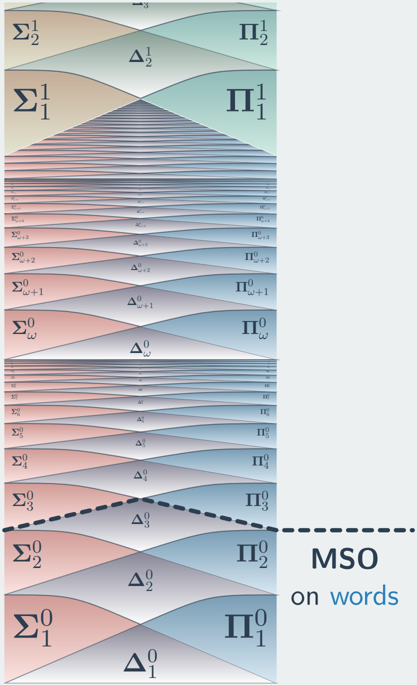

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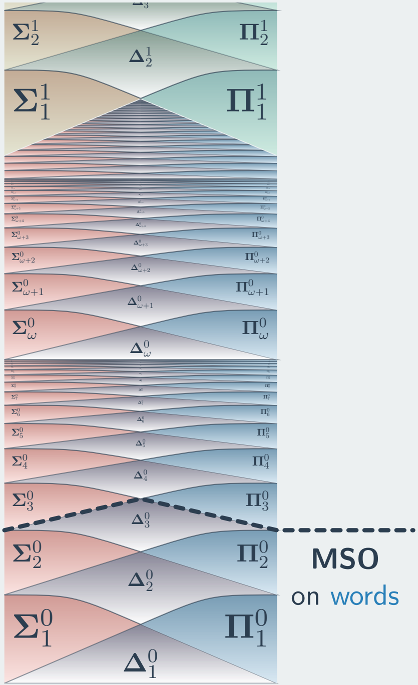
MSO on trees $\equiv$ non-det. aut. $\mathcal{A}$



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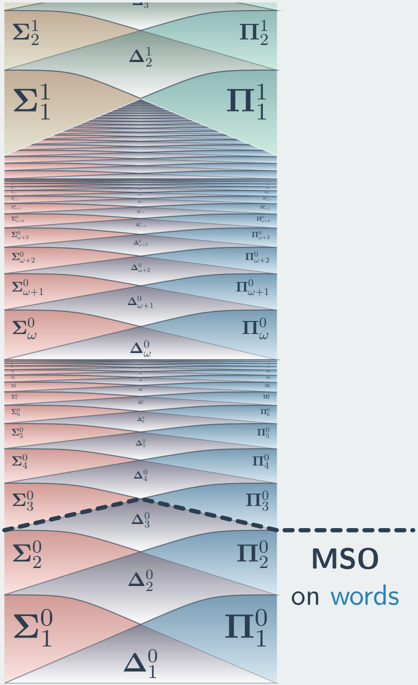

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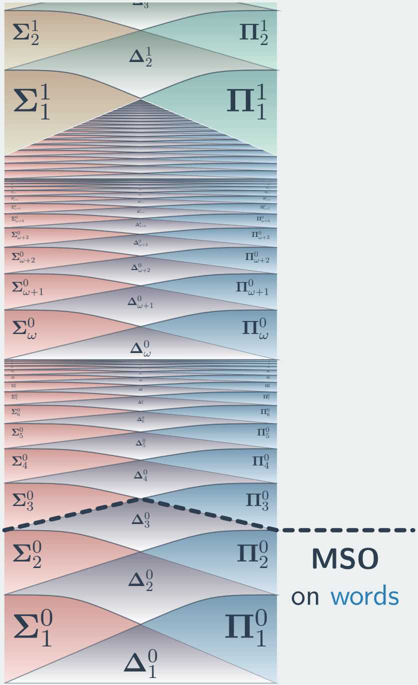
$$\rho \text{ is accepting on } \pi \}$$



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$$= \bigcap_x \bigcup_{y>x} \{\ldots\} \ \in \mathbf{\Pi}^0_2$$

MSO on trees $\equiv$ non-det. aut. $\mathcal{A}$

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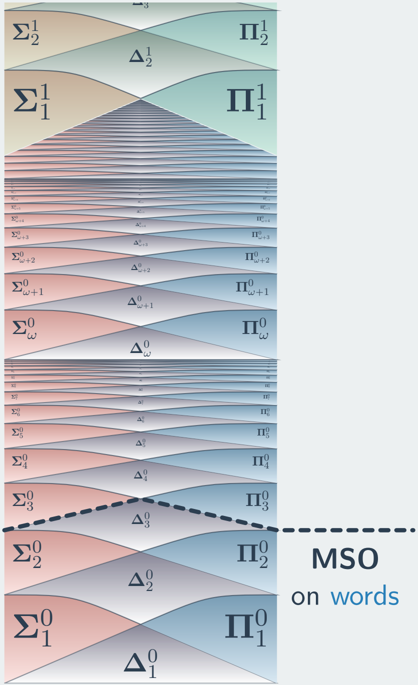

Upper bounds

$\exists X \rightsquigarrow$ projection $A \times \{0,1\} \to A$

$$\varphi = \exists X_1 \forall X_2 \ldots \exists X_n.\ \psi, \qquad \varphi \in \text{MSO}$$

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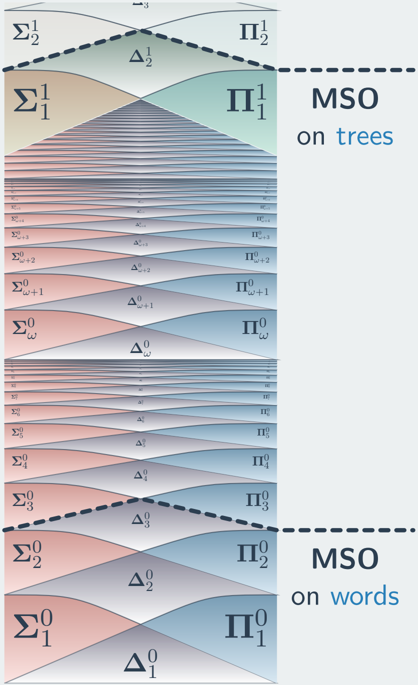

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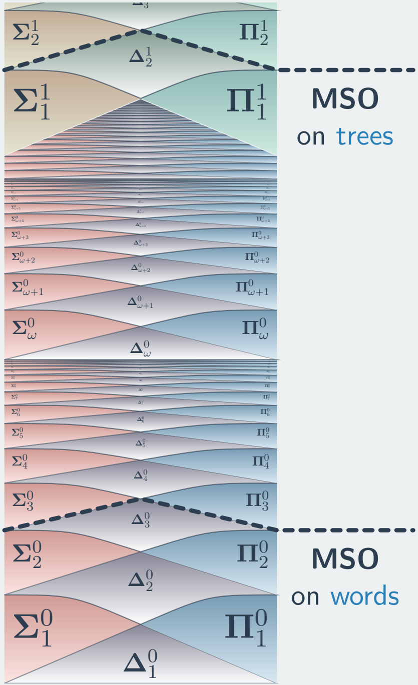

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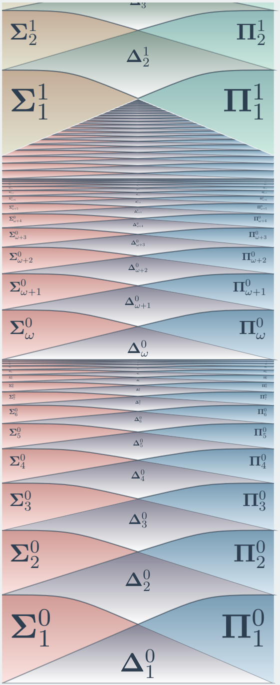

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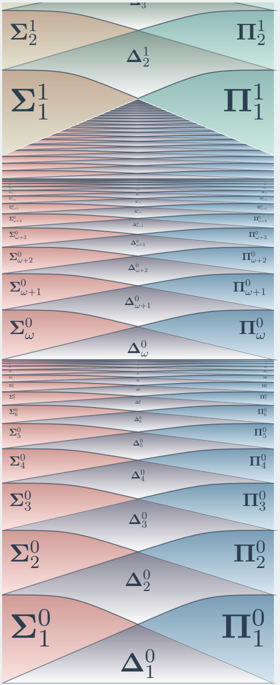
Theorem (Niwiński [1985])

There exists $\mathbf{\Sigma}^1_1$-complete (i.e. **non-Borel**) MSO-definable tree language.



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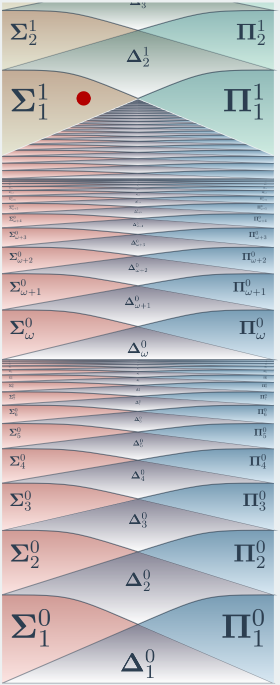

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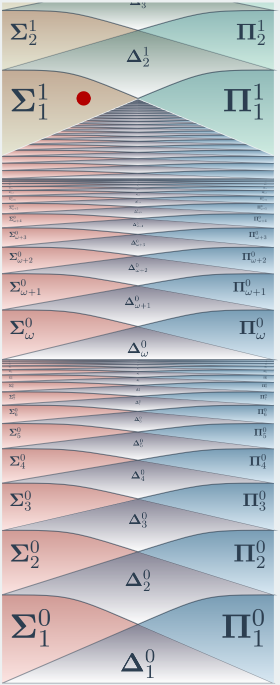

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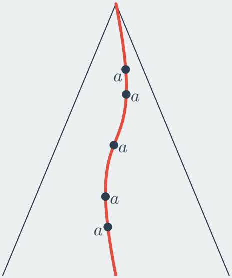

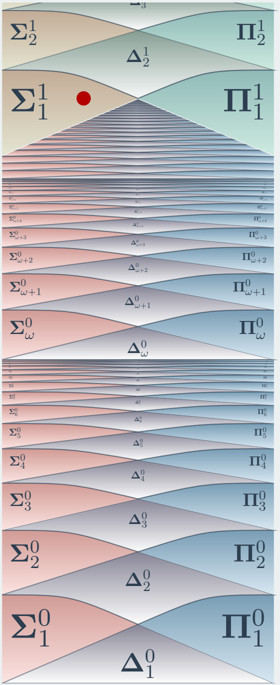

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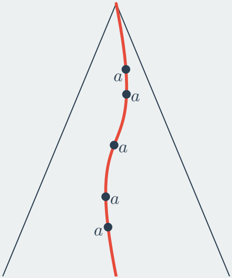

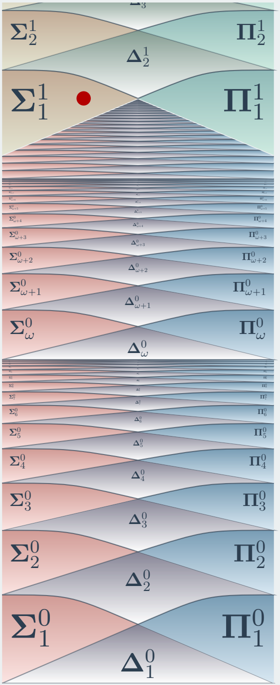

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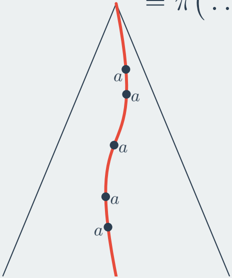

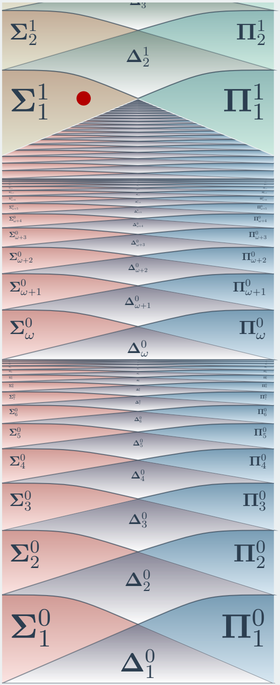

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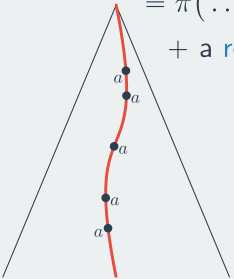

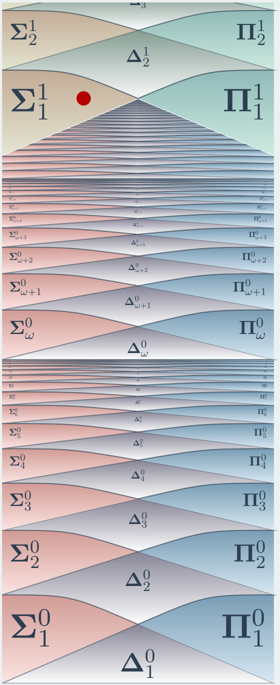

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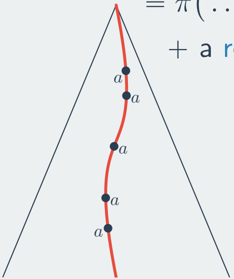

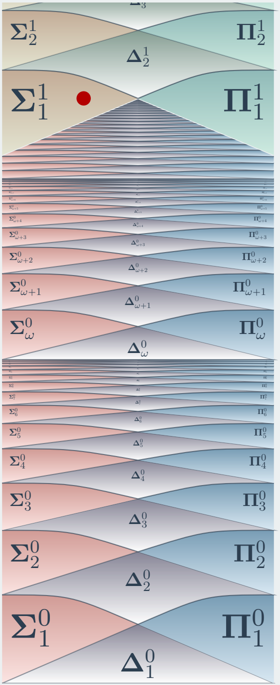

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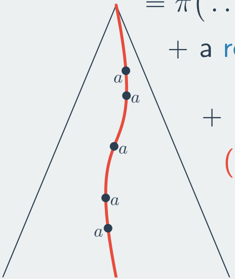

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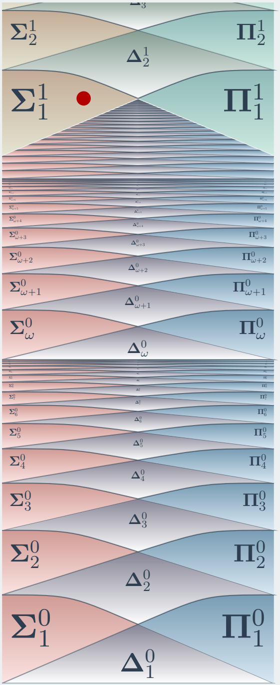

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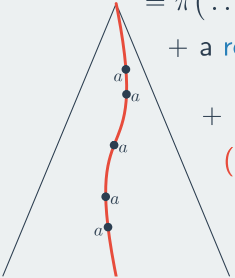

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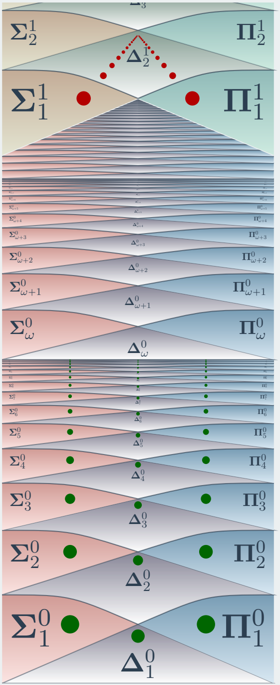

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$[\varphi \in \text{MSO}$ on trees$]$

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Part 1'

Topological complexity vs. decidability

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Large expressive power: cost functions, distance automata, ...

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⇝ no reasonable automaton model for MSO+U

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⇝ Further results (Bojańczyk et al. [2017]):

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Part 2

Reverse mathematics

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BUT: No third-order objects (like languages. . .)

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Rule of thumb: RCA_0 proves everything about finite combinatorics

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(**reverse** the implication)

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There is no arithmetic definition of truth.

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Solution:

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- formulate decidability of depth-n fragments for specific n
- study these sentences

Part 2.a

Reversing Büchi

(Kołodziejczyk, Michalewski, Pradic, S. [2016])

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[Ramsey's Theorem for Pairs]

$\mathrm{RT}^2_{<\infty}$

$\Rightarrow$

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[Ramsey's Theorem for Pairs] $\left\{ \begin{array}{l} \text{Every colouring of } [\omega]^2 \\ \quad \text{has infinite monochromatic set} \end{array} \right.$

$\mathrm{RT}^2_{<\infty}$

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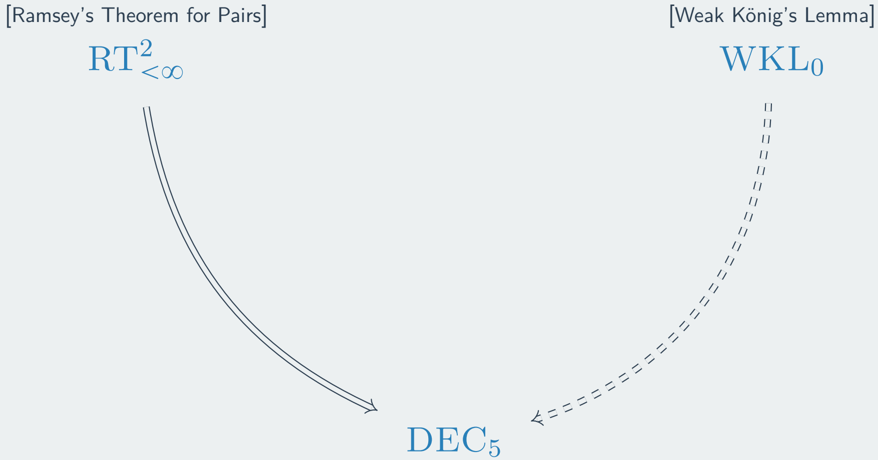

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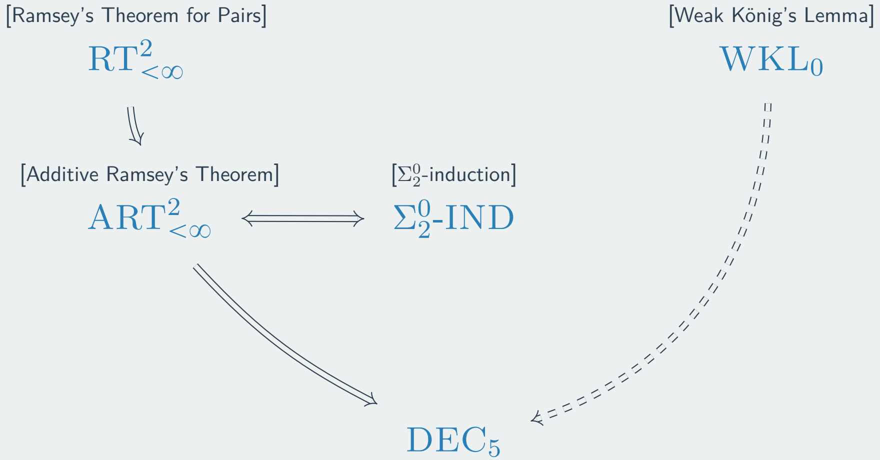

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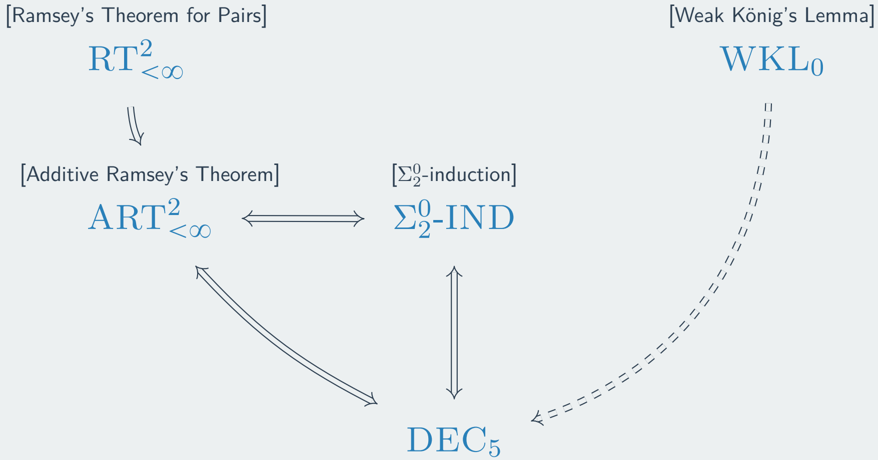

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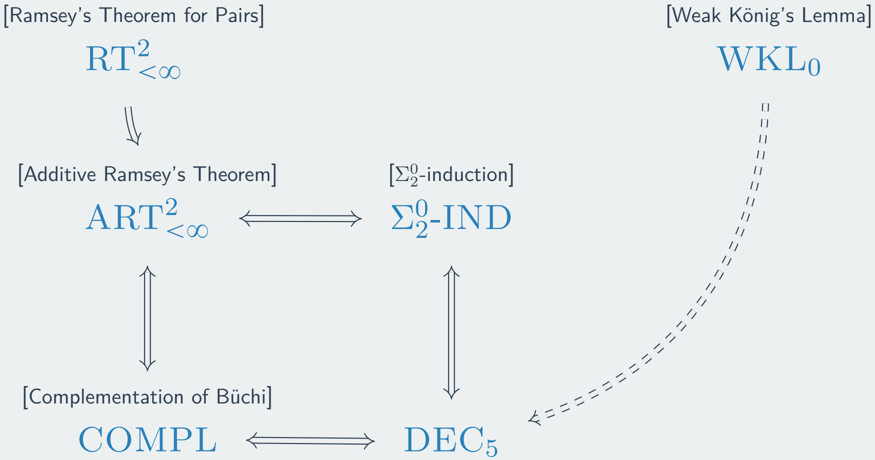

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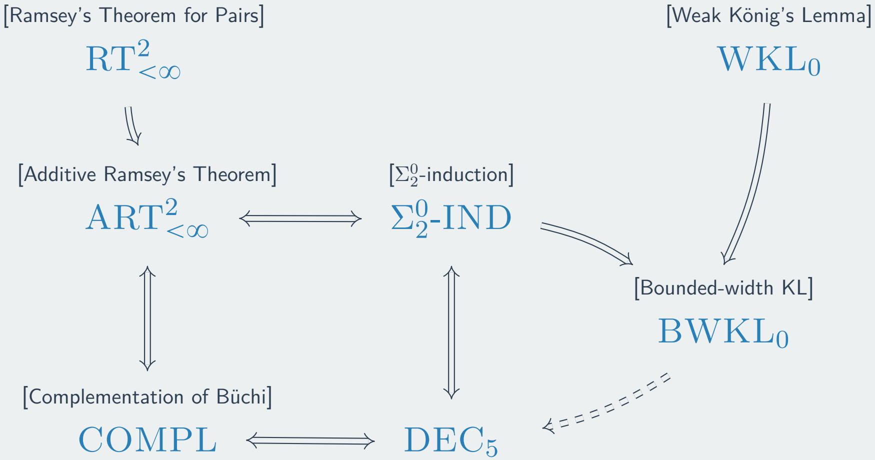

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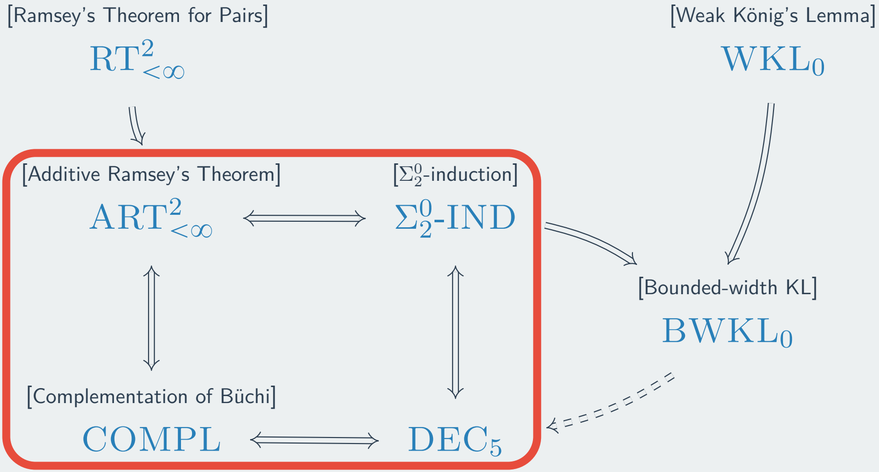

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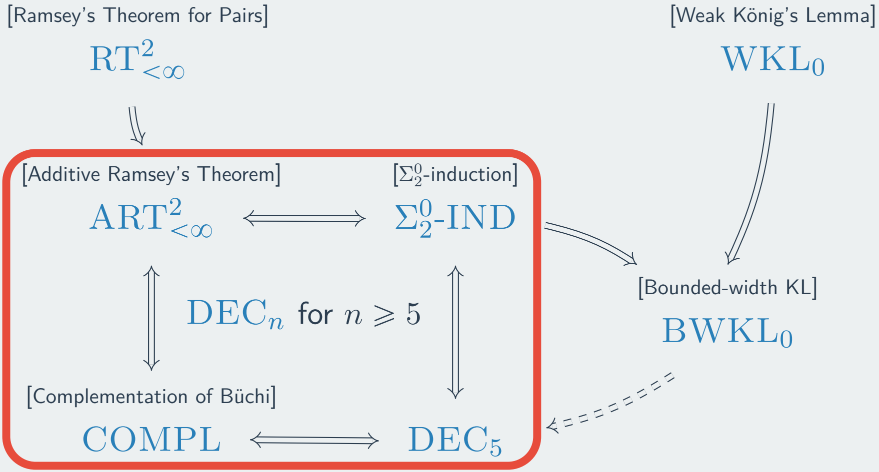

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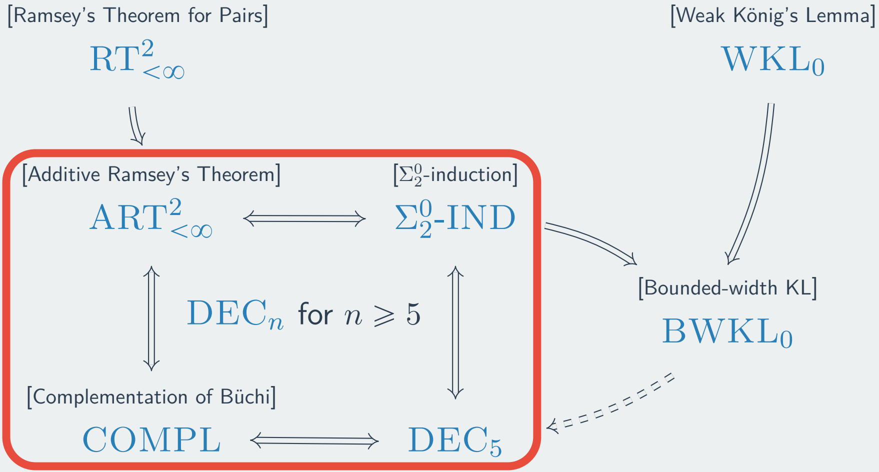

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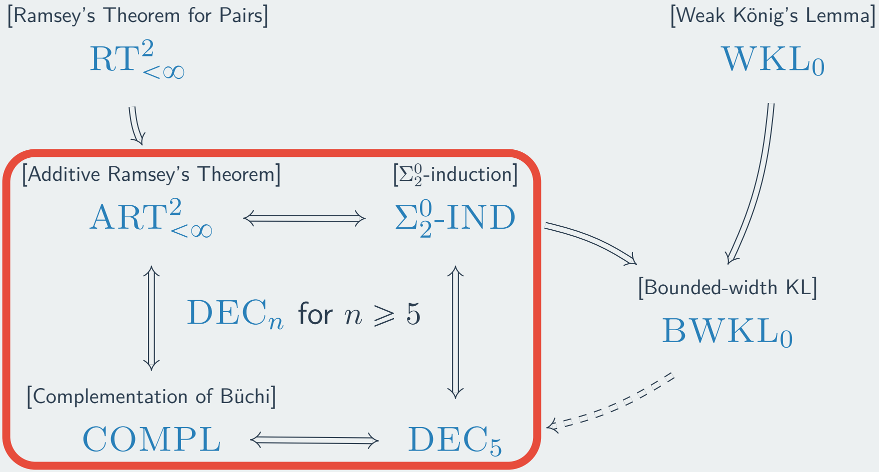

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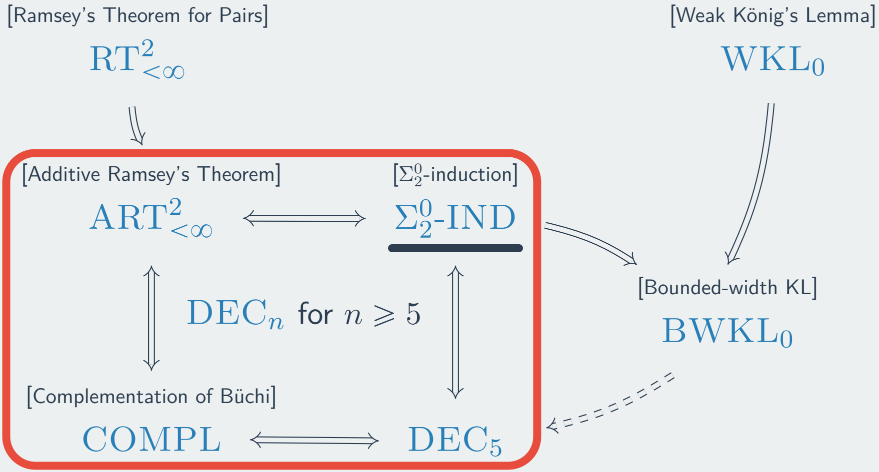

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(McNaughton and Safra constructions)

Part 2.b

Reversing Rabin

(Kołodziejczyk, Michalewski [2016])

Theorem (Rabin [1969])

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  then $\mathrm{RCA}_0 \vdash \big(\mathrm{COMPL} \iff \boldsymbol{\Psi}\big)$

(where COMPL is Rabin's complementation)

Part 2.b

Reversing Shelah

(Kołodziejczyk, Michalewski, Pradic, S. [2018?])

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