

Unambiguous Büchi is weak

Henryk Michalewski

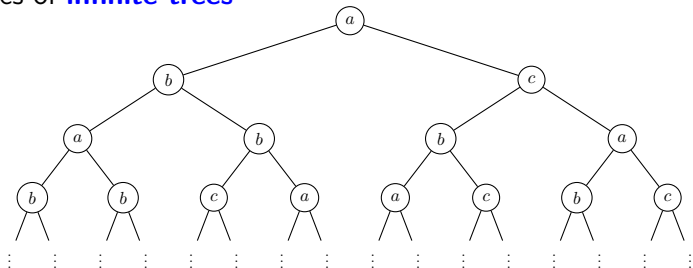
Michał Skrzypczak

DLT 2016

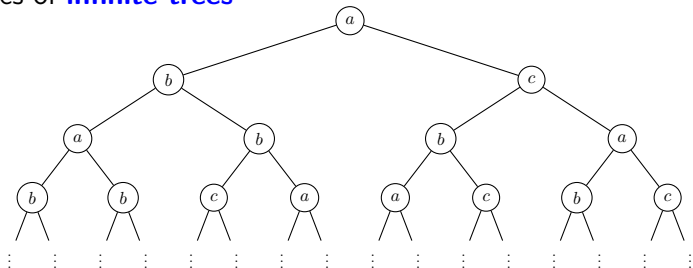
Montreal

Languages of **infinite trees**

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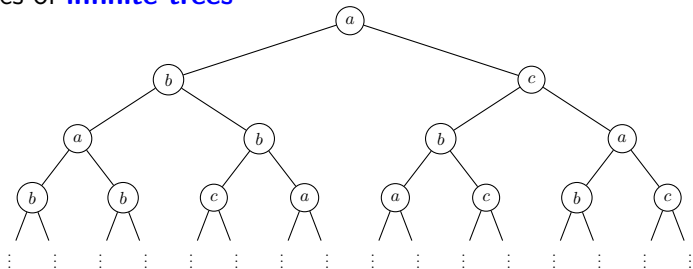
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Definable in **Monadic Second-Order logic (MSO)**

$$\neg, \vee, \exists x, \exists X, x \in X, x = y$$

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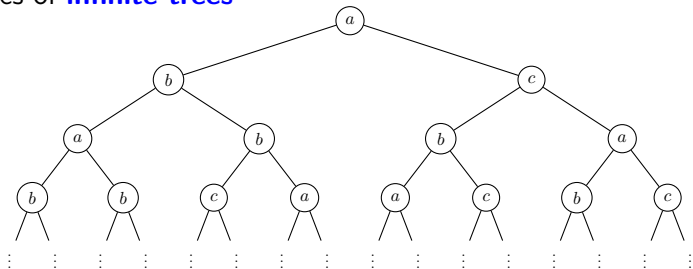
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The **MSO theory** of $(\{L, R\}^*, S_L, S_R)$ is **decidable**.

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Proof

Automata...

Parity automata

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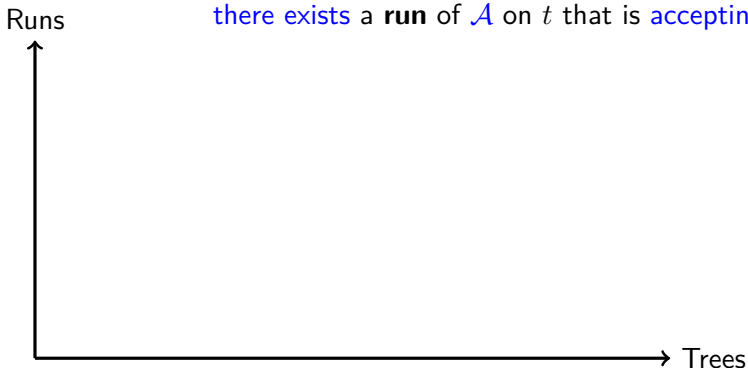
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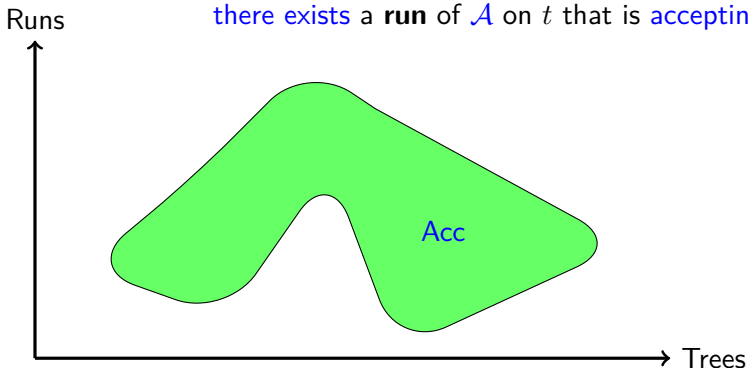
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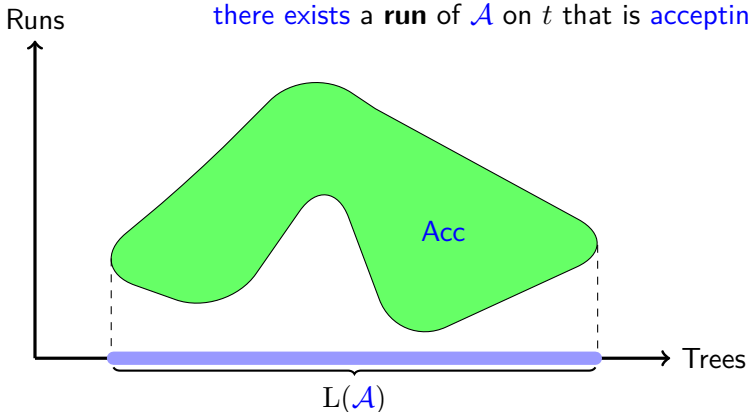
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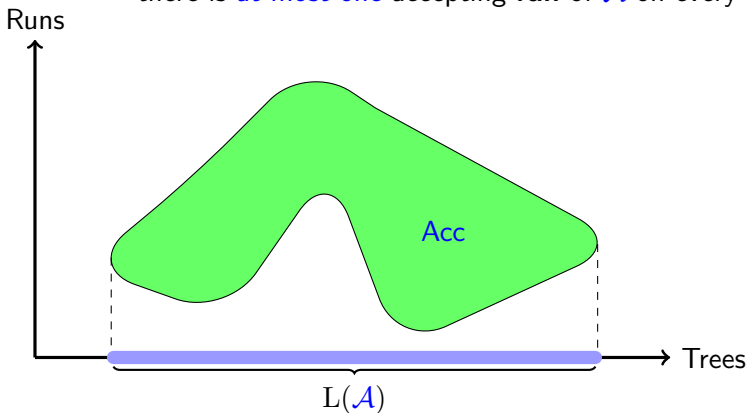
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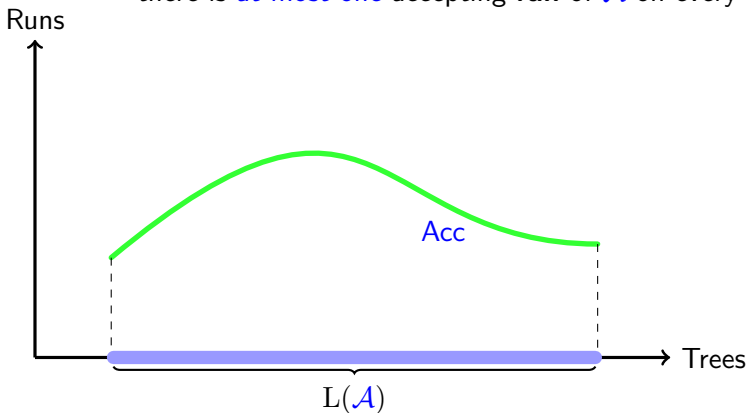
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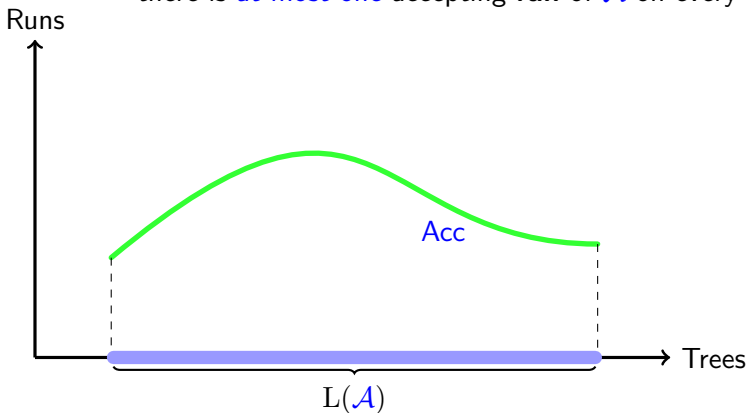
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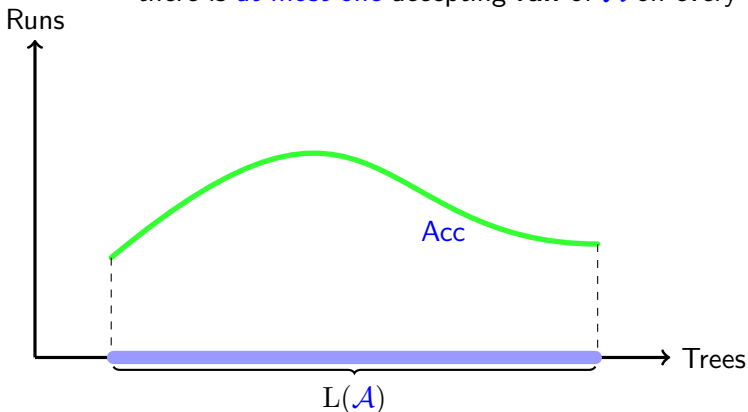
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(e.g. Stearns, Hunt [1985])

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→ **Goal:** understand **unambiguous** languages of **infinite trees**

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↳ **this work**

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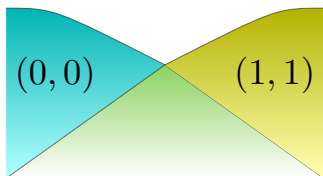
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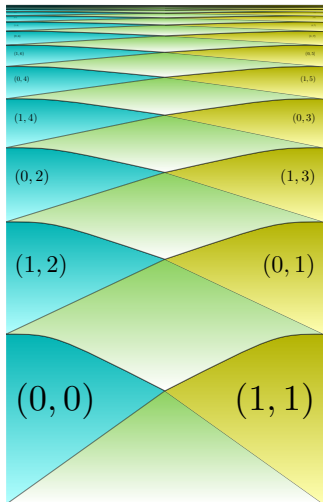
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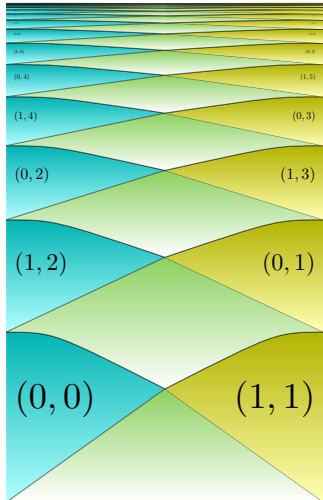
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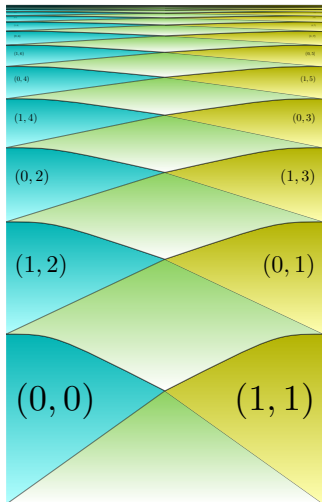
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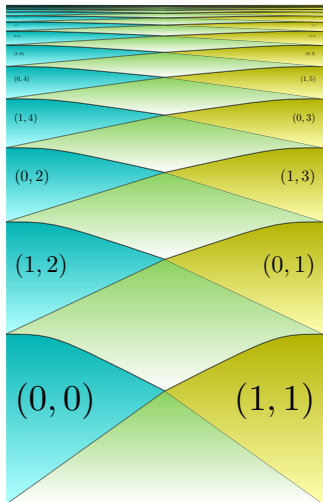
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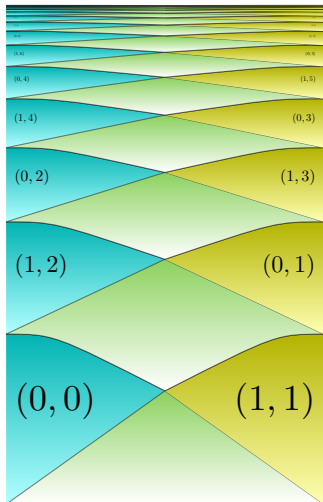
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(i.e. the hierarchies are **strict**)



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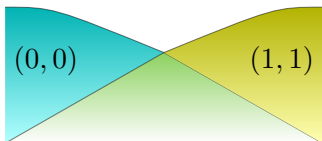
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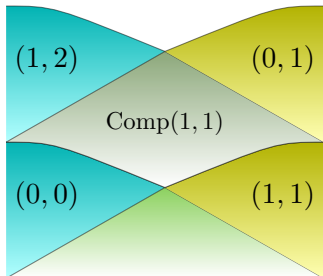
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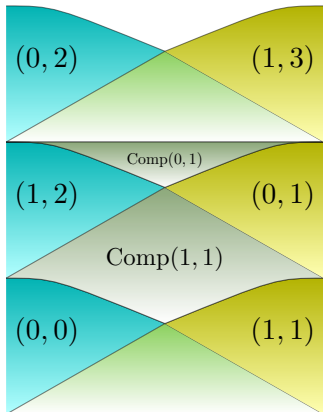
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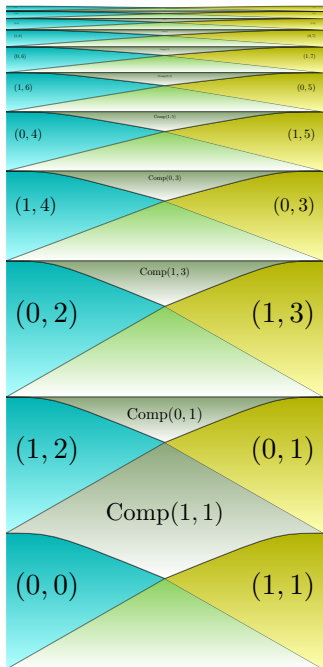
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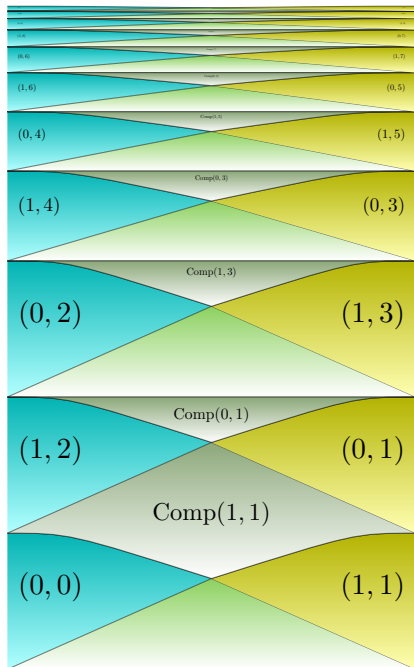
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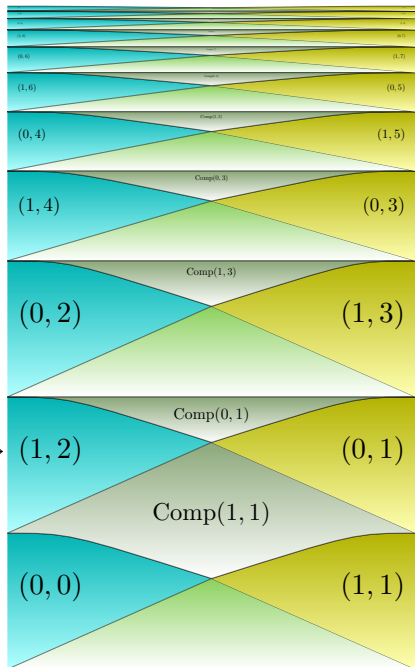


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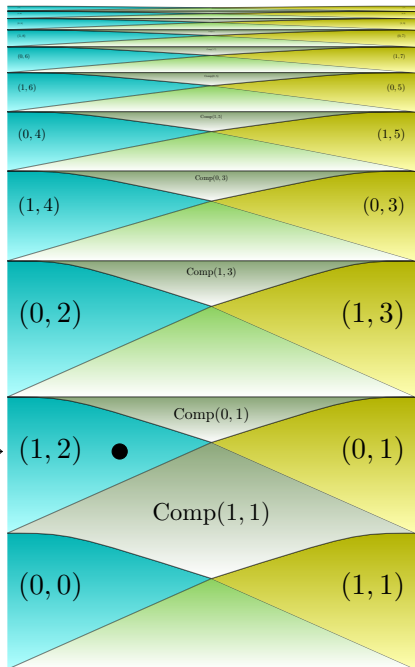
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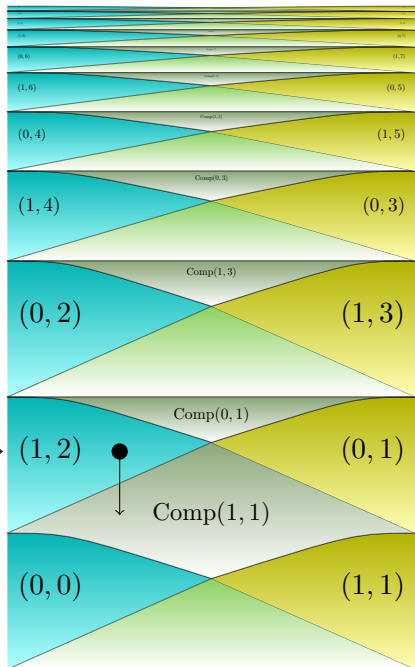
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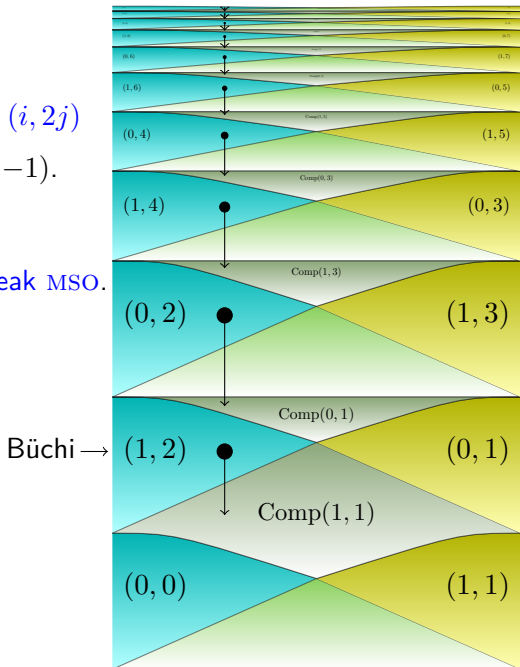
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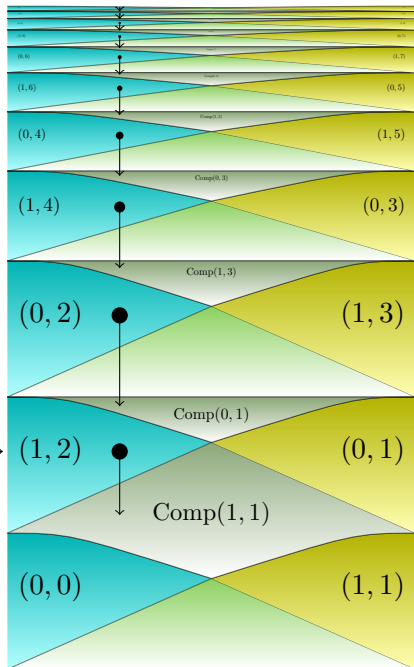
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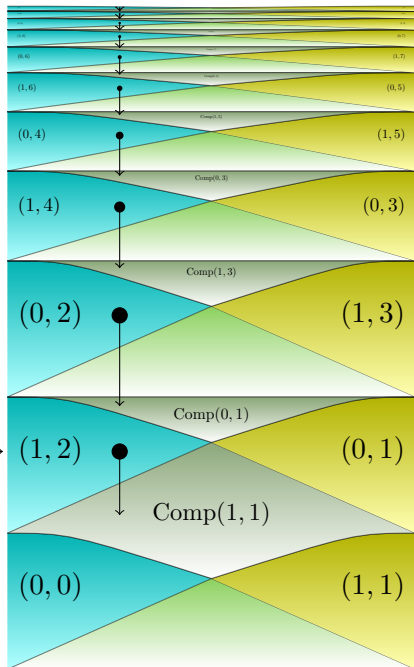
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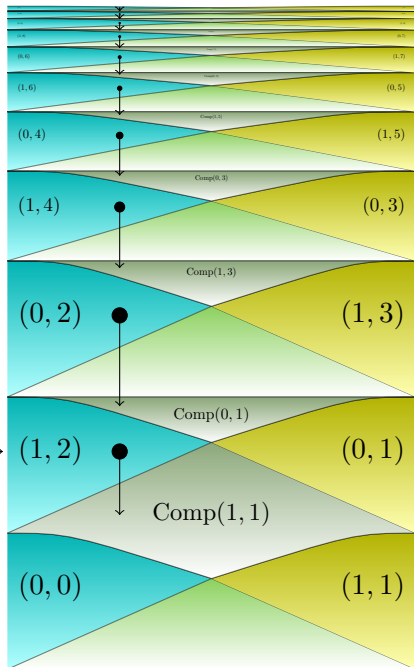
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