Deciding the topological complexity of Büchi languages

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Logics

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Monadic Second-Order (MSO) logic:

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- $\vee$, $\wedge$, $\neg$ (Boolean connectives)

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Weak Monadic Second-Order (WMSO) logic:

- only $\exists_x$ and $\exists^{\mathsf{fin}}_X$ quantifiers

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Monadic Second-Order ($\textsc{mso}$) logic:

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Existential Monadic Second-Order ($\exists$MSO) logic:

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Other formalisms: LTL, CTL*, modal μ-calculus, ...

Decidability: infinite trees

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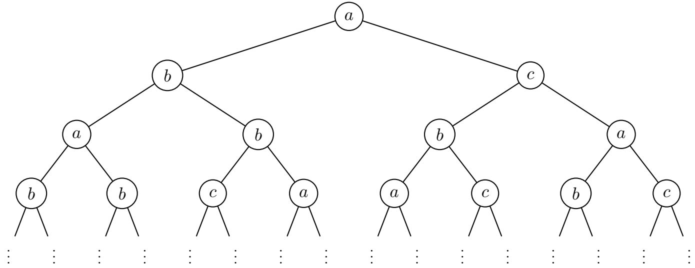
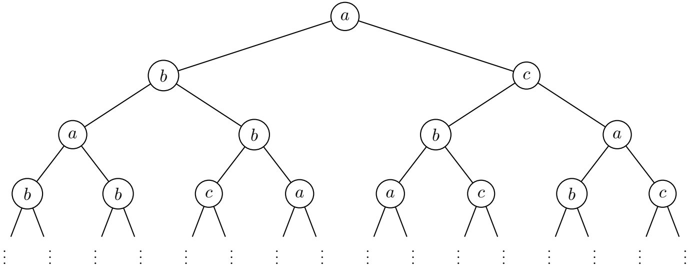


Decidability: infinite trees



$$t\colon \{\mathtt{L},\mathtt{R}\}^* \to A$$

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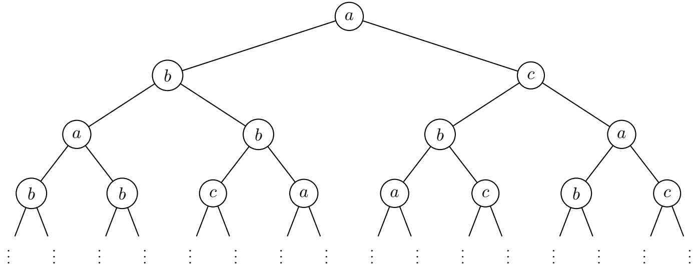

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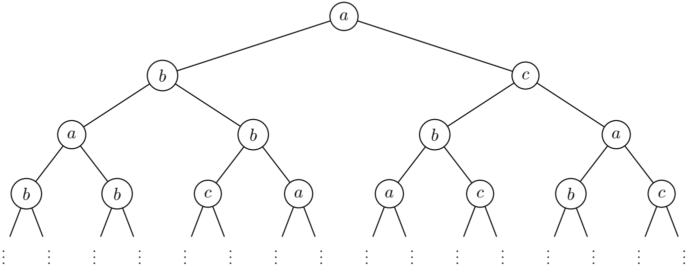

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Theorem (Rabin 1969)

The MSO theory of $\big(\{\mathtt{L}, \mathtt{R}\}^*, \mathrm{s}_{\mathtt{L}}, \mathrm{s}_{\mathtt{R}}\big)$ is decidable.

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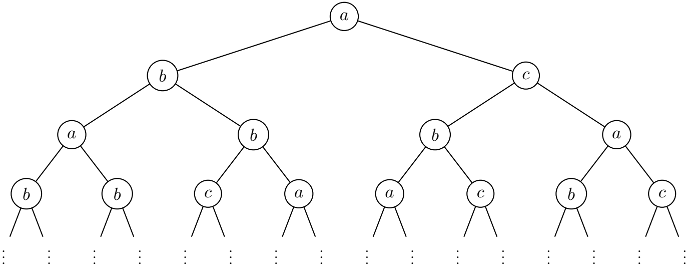

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☺ **"the mother of all decidability results"** ☺

Expressive power

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WMSO

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Logics: Quantifiers:

WMSO $\exists_x$, $\exists^{\mathsf{fin}}_X$

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Expressive power

Logics:	Quantifiers:	Automata:
WMSO	$\exists_x$, $\exists_X^{\mathsf{fin}}$	weak alternating
$\exists$MSO	$\exists_{X_1} \ldots \exists_{X_n}\ \psi$ for $\psi \in$ WMSO	
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WMSO	$\exists_x$, $\exists^{\mathsf{fin}}_X$	weak alternating
$\exists$MSO	$\exists_{X_1} \ldots \exists_{X_n}\ \psi$ for $\psi \in$ WMSO	non-deterministic Büchi alternating Büchi
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Question: are these containments strict?

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Question: are these containments strict? (yes)

Expressive power

Logics:	Quantifiers:	Automata:
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Input: φ from a stronger logic $\mathcal{L}$

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Question: are these containments strict? (yes)

Problem: characterise these logics (effective characterisation)

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Effective characterisations and expressive power

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Schutzenberger, McNaughton, Papert,

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Place, Zeitoun, . . .

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Theorem (Colcombet, Kuperberg, Löding, Vanden Boom [2013])
It is decidable if a given $\exists$MSO formula is equivalent to a WMSO formula.

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⇝ not much insight about WMSO

Topology of infinite trees

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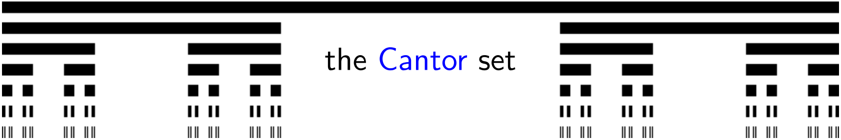

the Cantor set

Descriptive set theory

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Start from simple sets

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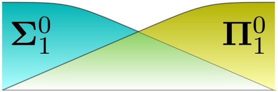


Descriptive set theory

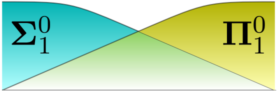
Start from simple sets

— open ($\mathbf{\Sigma}^0_1$) and closed ($\mathbf{\Pi}^0_1$)

Apply countable unions ($\bigcup$)

and countable intersections ($\bigcap$)

— $\mathbf{\Sigma}^0_{\eta+1}$ and $\mathbf{\Pi}^0_{\eta+1}$



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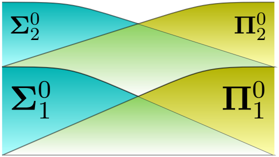

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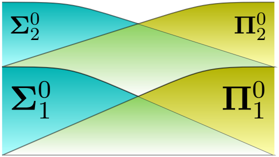
By induction



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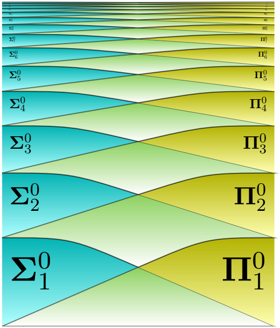

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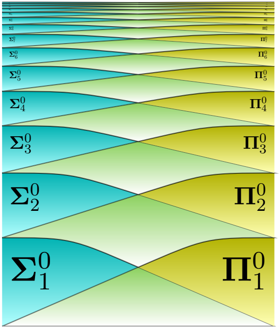

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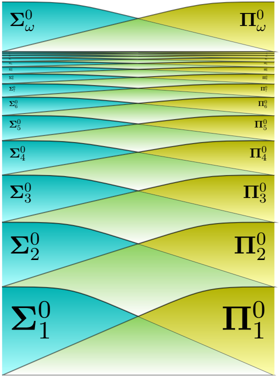

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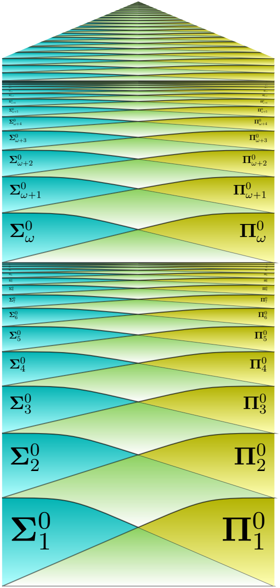

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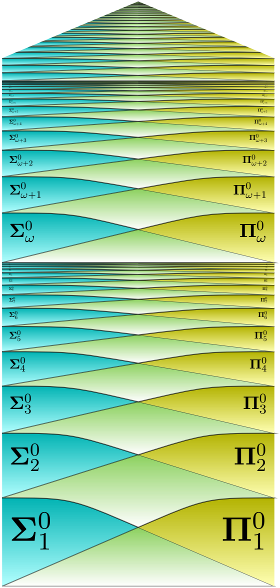

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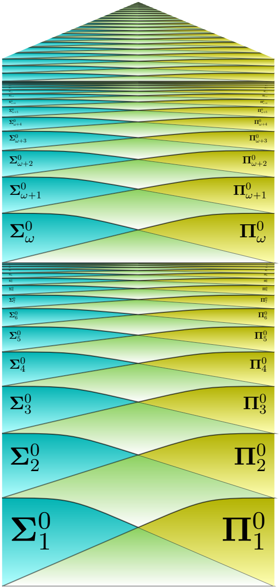
Apply projection and co-projection

— analytic ($\mathbf{\Sigma}^1_1$) and co-analytic ($\mathbf{\Pi}^1_1$)



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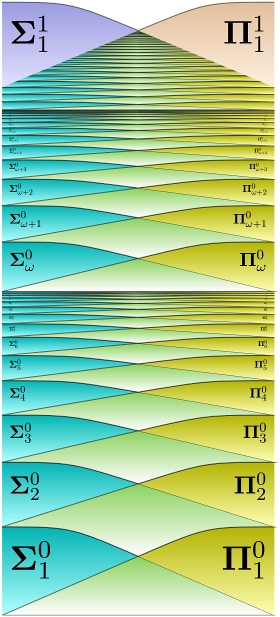

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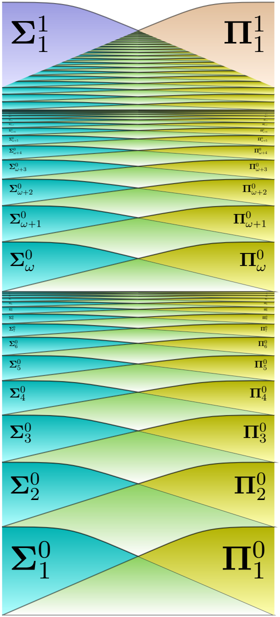

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— $\mathbf{\Sigma}^0_{\eta+1}$ and $\mathbf{\Pi}^0_{\eta+1}$

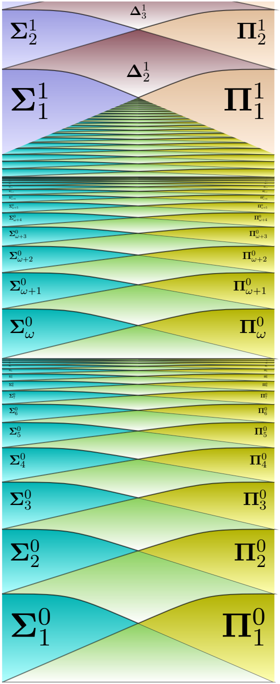
By induction (transfinite)

— Borel sets: $\mathbf{\Sigma}^0_\eta$, $\mathbf{\Pi}^0_\eta$ for $\eta < \omega_1$

Apply projection and co-projection

— analytic ($\mathbf{\Sigma}^1_1$) and co-analytic ($\mathbf{\Pi}^1_1$)

By induction



Descriptive set theory

Start from simple sets

— open ($\mathbf{\Sigma}^0_1$) and closed ($\mathbf{\Pi}^0_1$)

Apply countable unions ($\bigcup$)

and countable intersections ($\bigcap$)

— $\mathbf{\Sigma}^0_{\eta+1}$ and $\mathbf{\Pi}^0_{\eta+1}$

By induction (transfinite)

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Apply projection and co-projection

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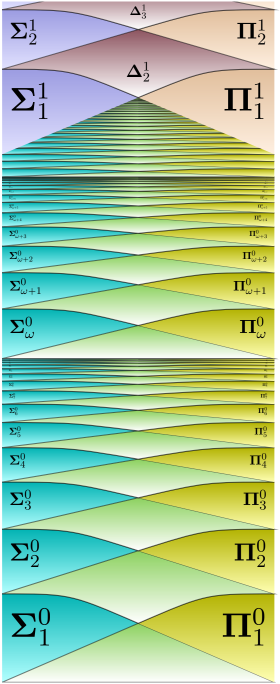
By induction

— projective sets: $\mathbf{\Sigma}^1_n$, $\mathbf{\Pi}^1_n$ for $n < \omega$



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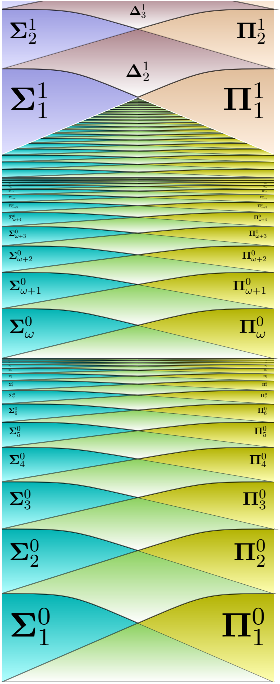

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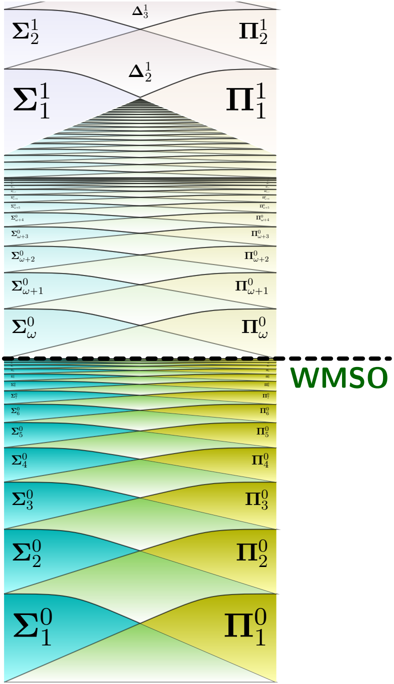
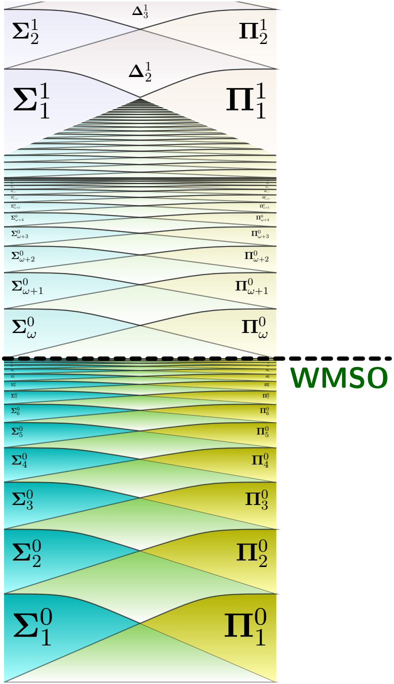


Upper bounds

$$L \in \mathrm{WMSO} \implies L \in \boldsymbol{\Sigma}^0_n \ \text{(for some } n\text{)}$$



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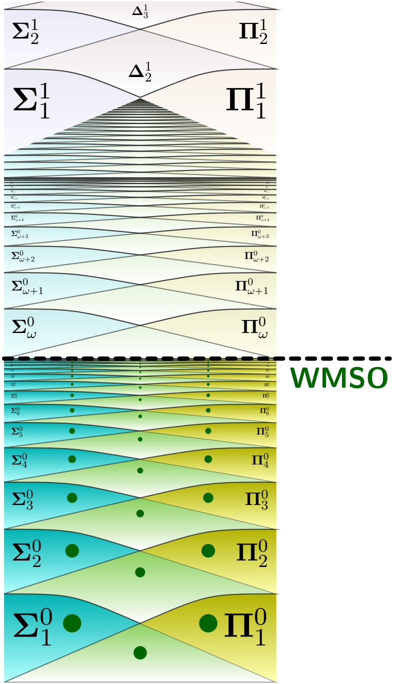

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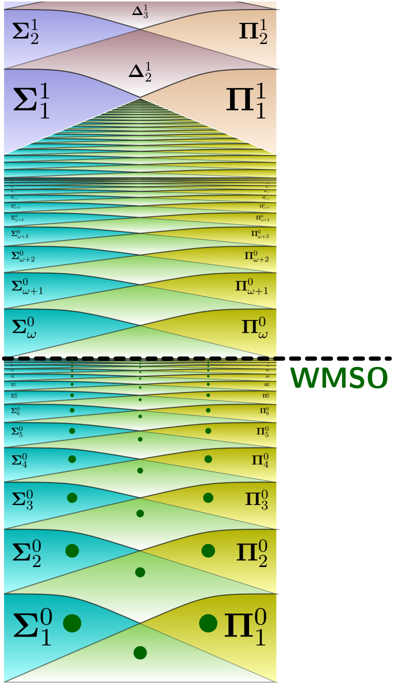

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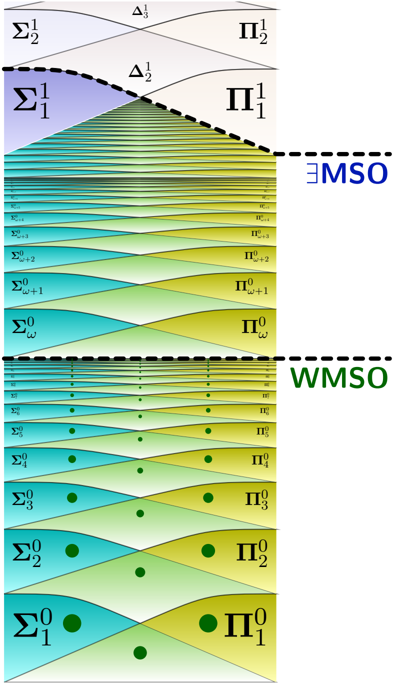

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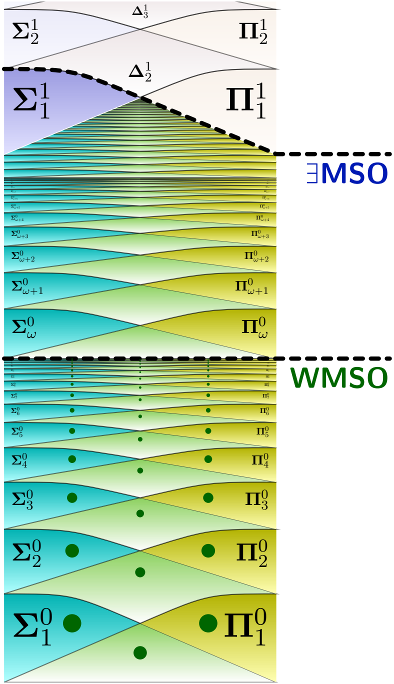

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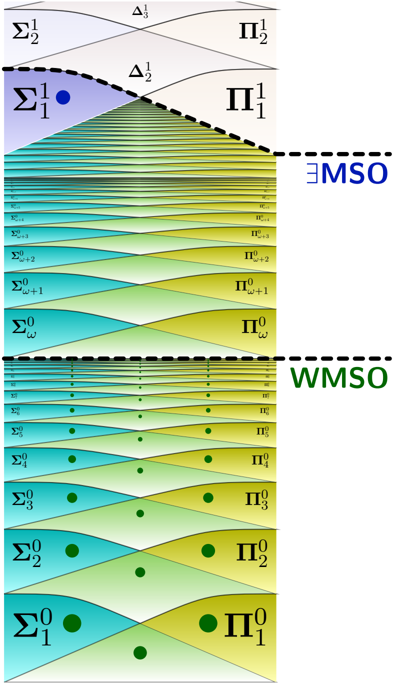

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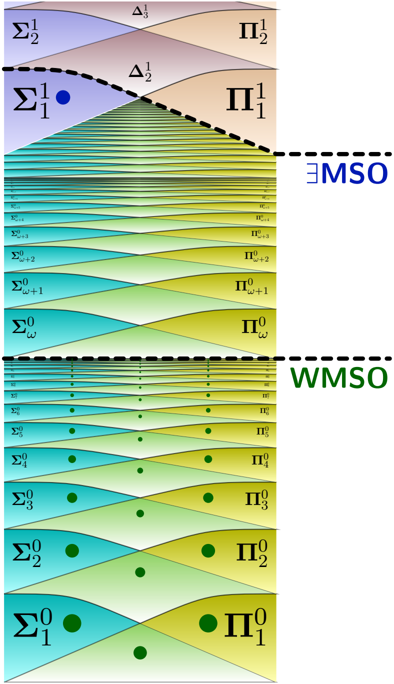

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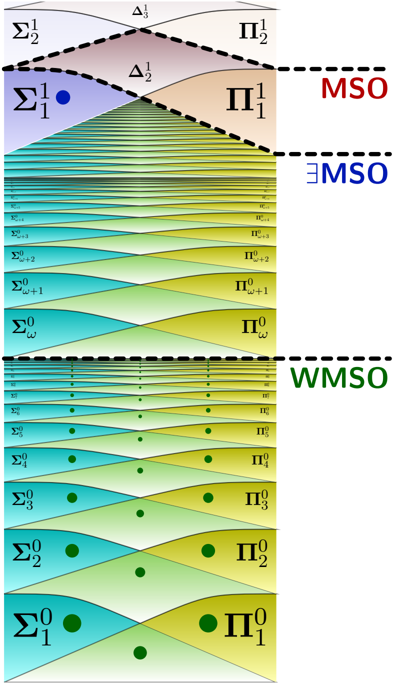

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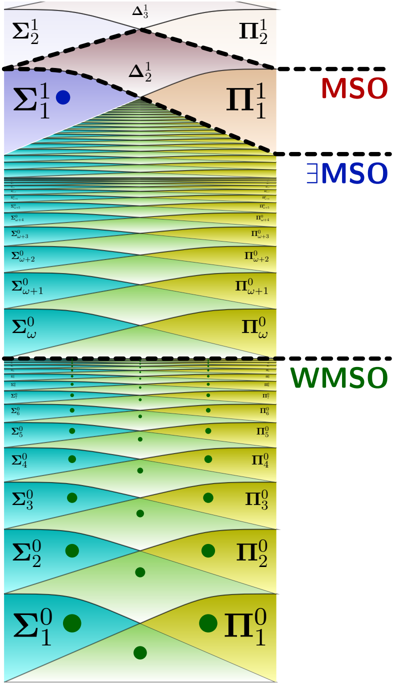

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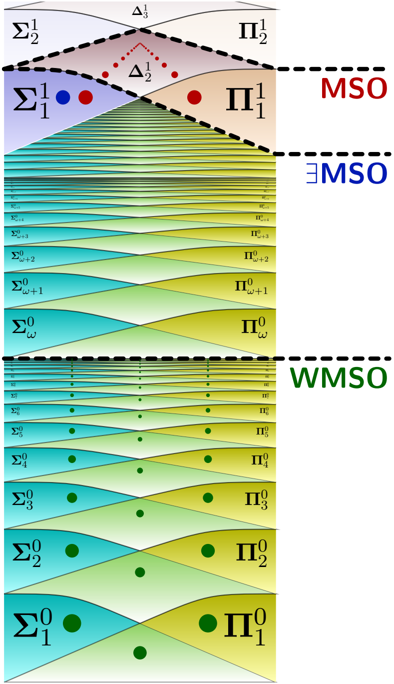

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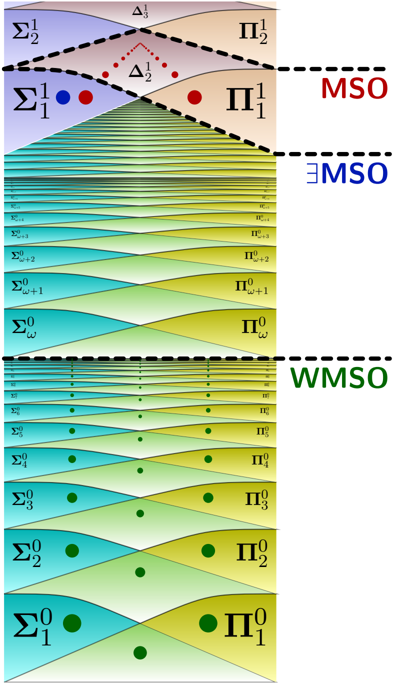

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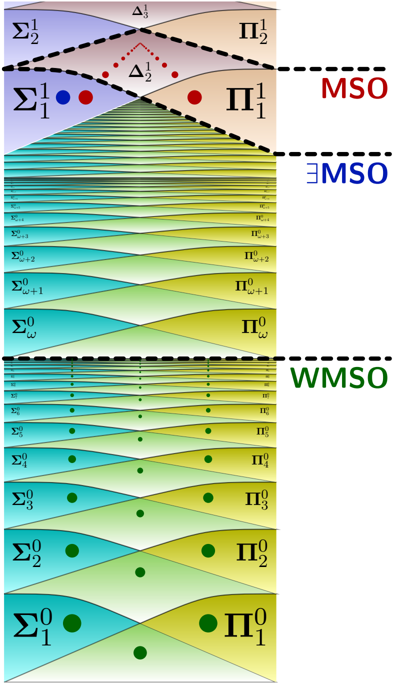

Gap property

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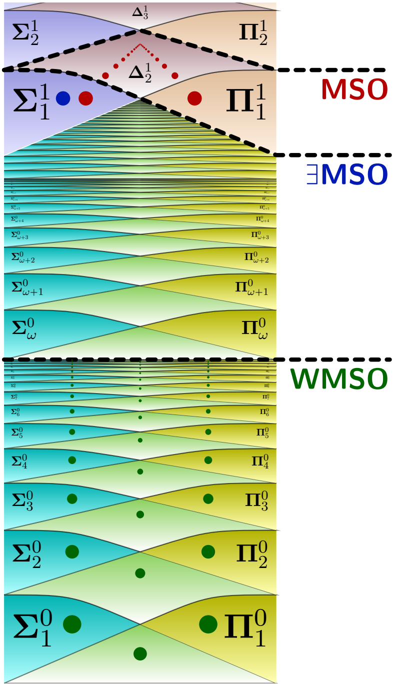

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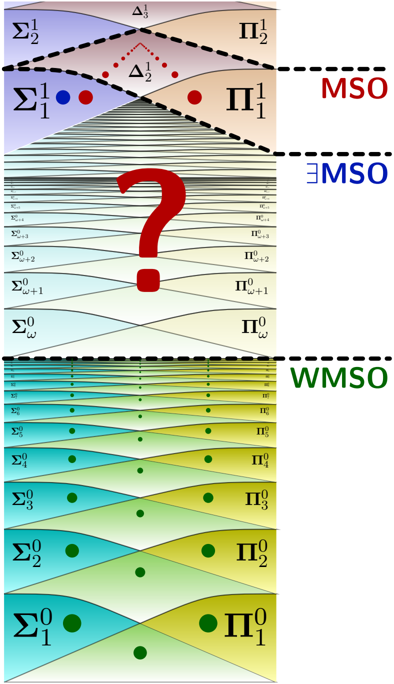
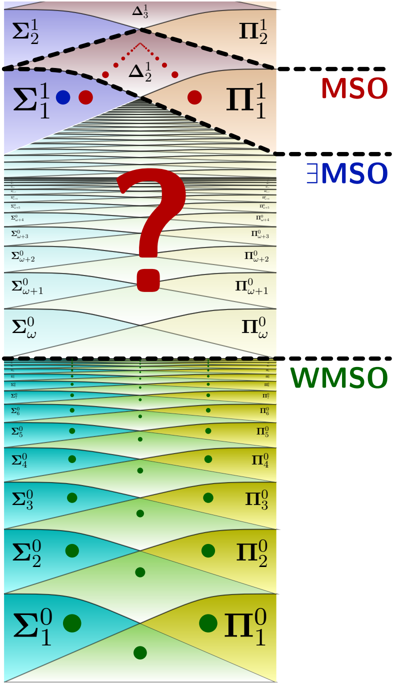


Gap property

Conjecture (Skurczyński [1993])

A Borel MSO-definable language is WMSO-definable.



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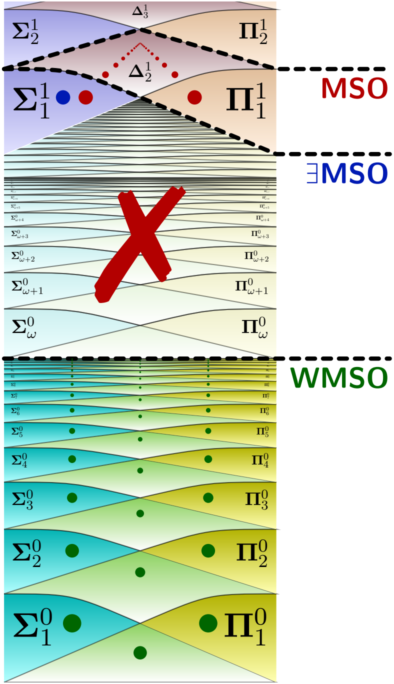

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An ∃MSO-definable language is either:

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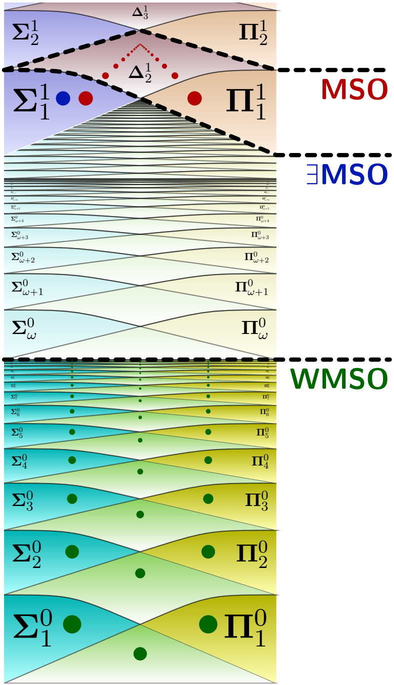

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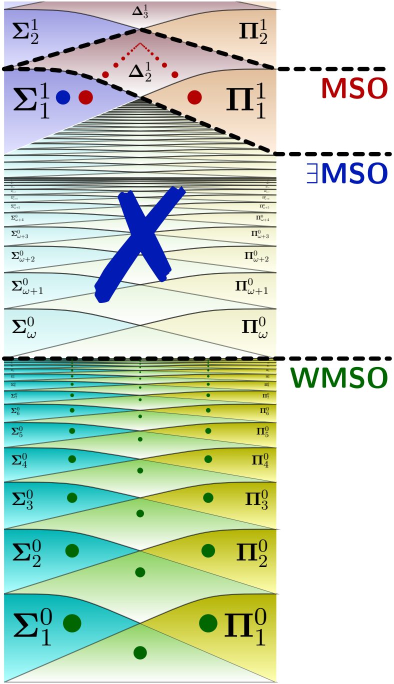

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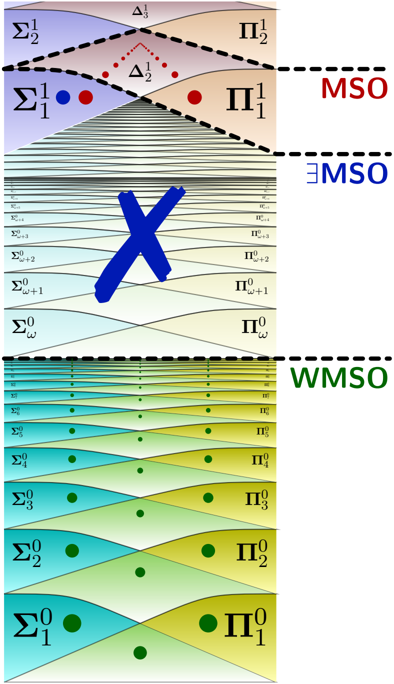

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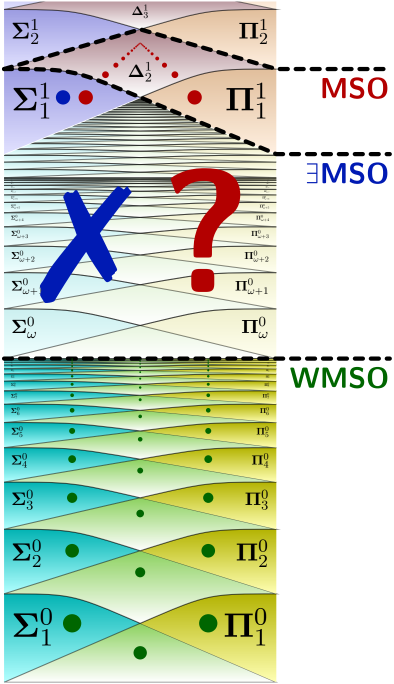

Theorem (Walukiewicz, S. [2016])

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Construct a finite game $\mathcal{F}$ with an ω-regular winning condition

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