On uniformisability in monadic second-order logic

Michał Skrzypczak

LIAFA, University of Warsaw

Uniformisation

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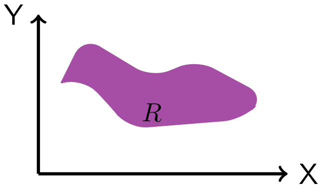

Relation $R \subseteq X \times Y$

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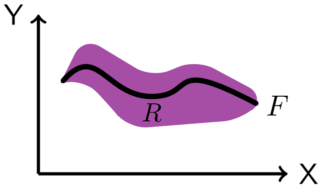

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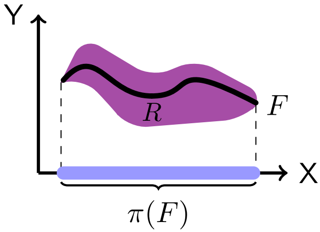

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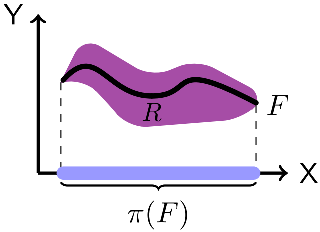

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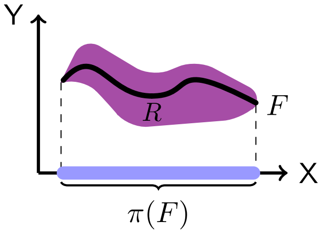

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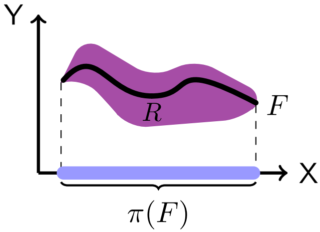

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Theorem (Novikov, Kondô [1938])

Every co-analytic ($\mathbf{\Pi}^1_1$) relation admits a co-analytic uniformisation.

Structures

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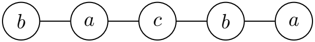
$w =$ 

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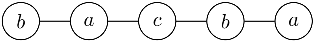
$w =$ 

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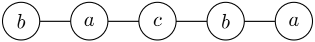
$w =$ 

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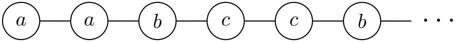
$\alpha =$ 

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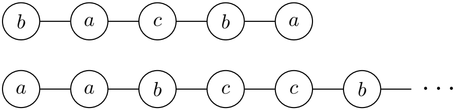

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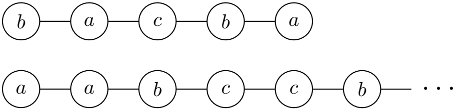
$w =$ 

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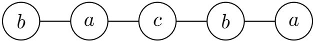
$w =$ 

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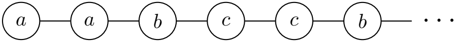
$\alpha =$ 

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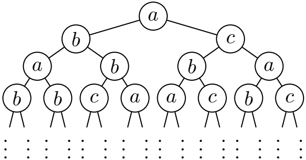
$t =$ 

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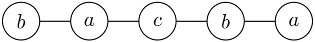
$w =$ 

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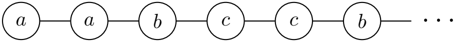
$\alpha =$ 

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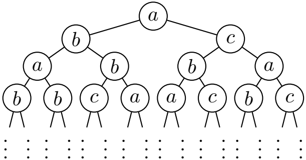
$t =$ 

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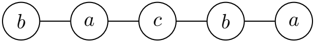
$w =$ 

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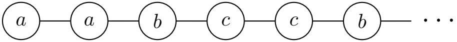
$\alpha =$ 

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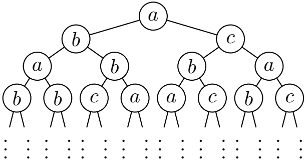
$t =$ 

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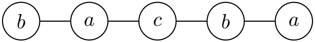
$w =$ 

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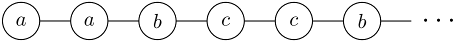
$\alpha =$ 

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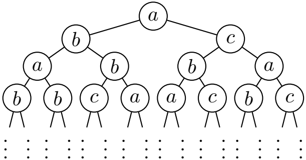
$t =$ 

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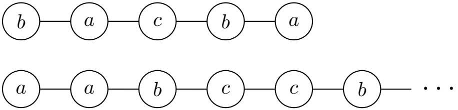

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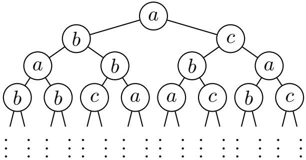

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$(s, s') \sim s \otimes s' = \begin{smallmatrix} a & b & c \\ x & y & z \end{smallmatrix}$ over $A \times A'$

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$\rightsquigarrow$ applications to verification and model-checking

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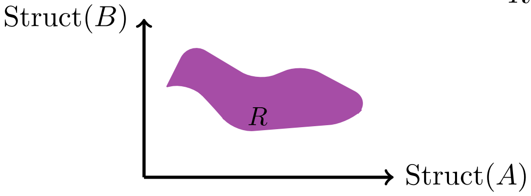

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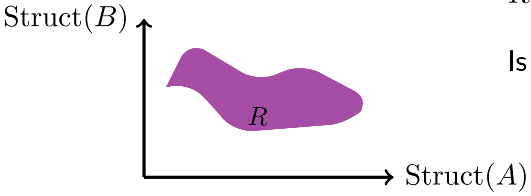

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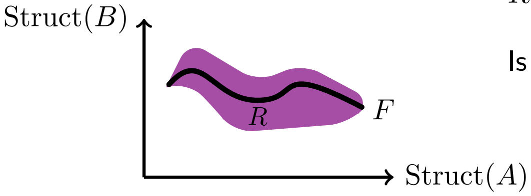

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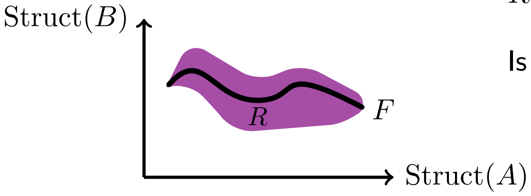

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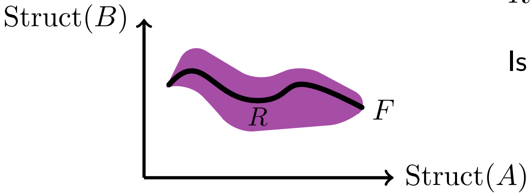

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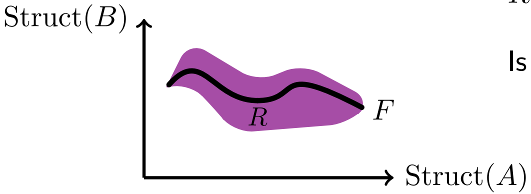

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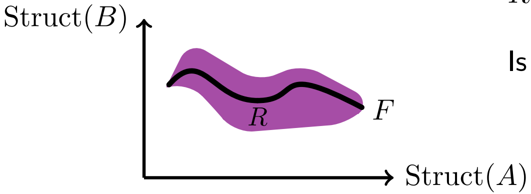

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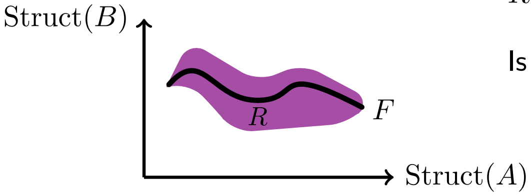

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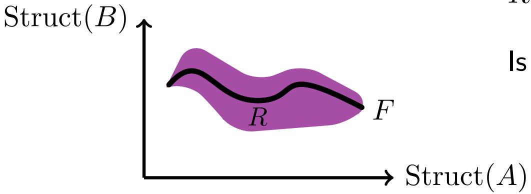

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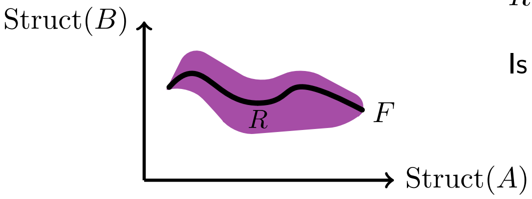

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— ψ can be effectively constructed from φ?

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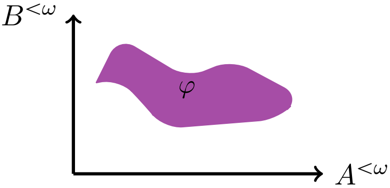

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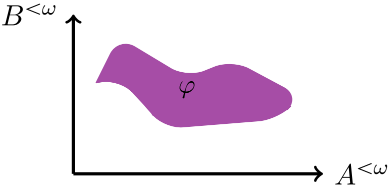

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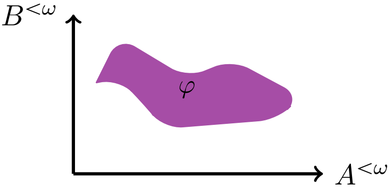

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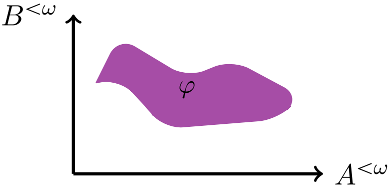
Take φ over $A \times B$

Let $(s, s') \in F$ if:

— $s \otimes s' \models \varphi$

— for every s'' such that $s \otimes s'' \models \varphi$



MSO **over finite words ✓**

Take φ over $A \times B$

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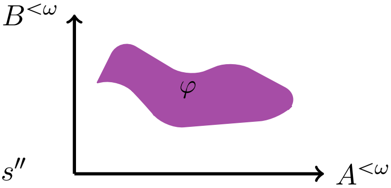

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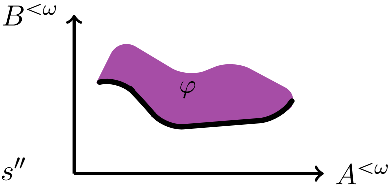

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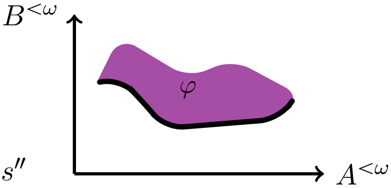

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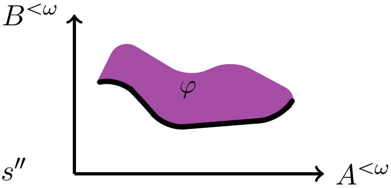

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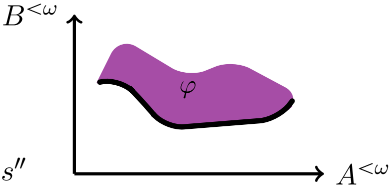

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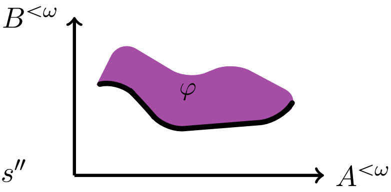

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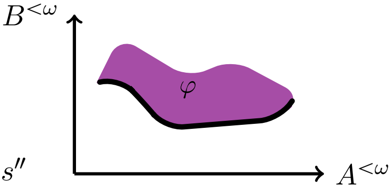

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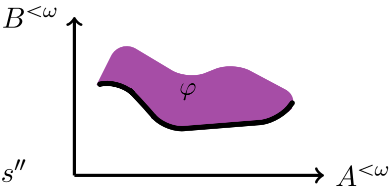

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$$(bbbb\ldots) \geqslant_{\text{lex}} (abbb\ldots) \geqslant_{\text{lex}} (aabb\ldots) \geqslant_{\text{lex}} (aaab\ldots) \geqslant_{\text{lex}} \ldots$$

MSO over infinite words ?

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Lifsches, Shelah, JSL [1998]

THEOREM 6.3. $(\omega, <)$ *has the uniformization property.*

PROOF. By [1]. ⊣

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$\Big[$both $\begin{smallmatrix} a & a & \sharp & a & a \\ ? & ? & \sharp & 0 & 1 \end{smallmatrix}$ and $\begin{smallmatrix} a & a & \sharp & a & a \\ 0 & 1 & \sharp & ? & ? \end{smallmatrix}$ belong to $R\Big]$

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⇝ **no** uniformisation in FO over **finite/infinite words/trees**

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Does MSO have uniformisation property over infinite trees?

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Proof

Pumping of runs of a marking automaton. ■

Parameters

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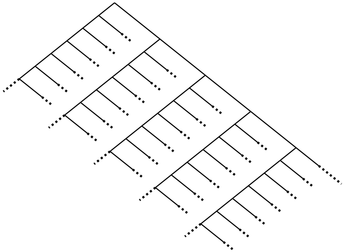

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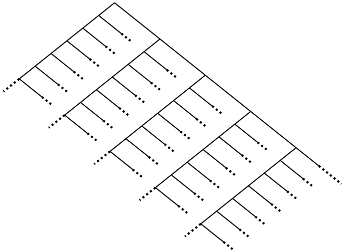

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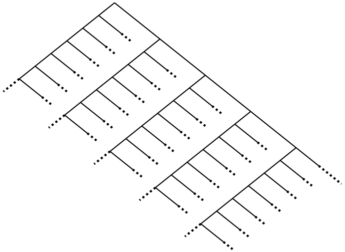

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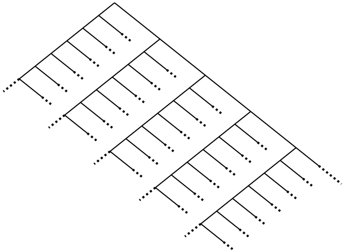

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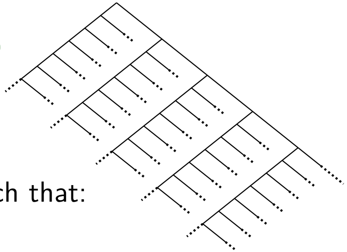

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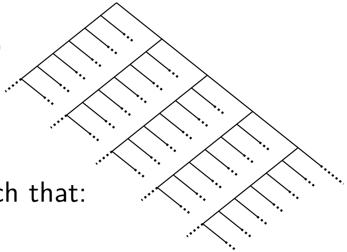

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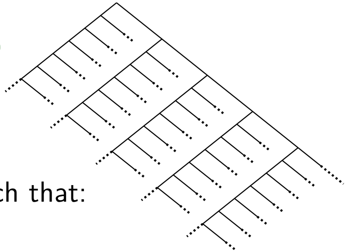

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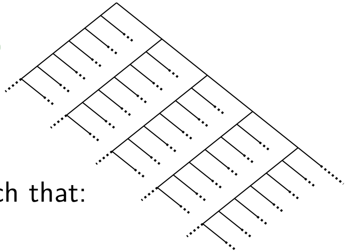

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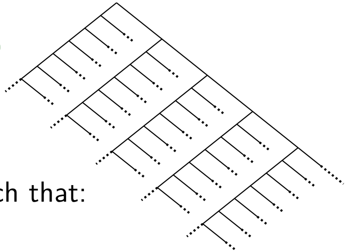

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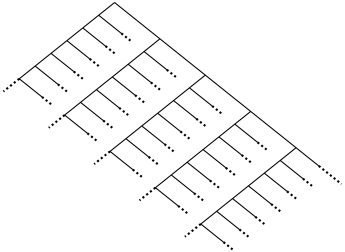

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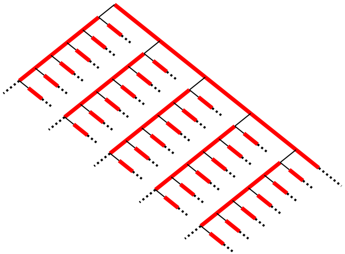

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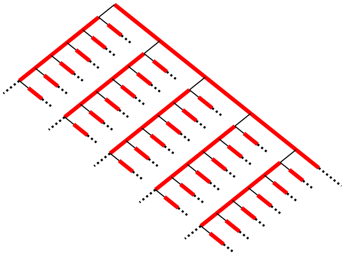

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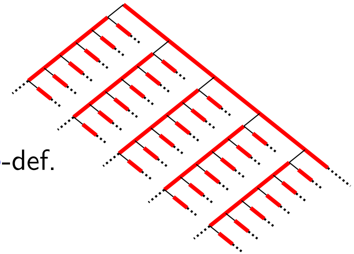

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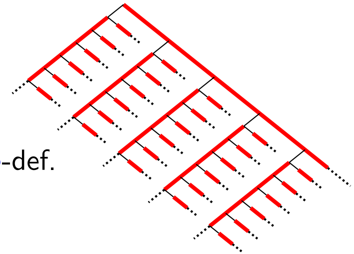

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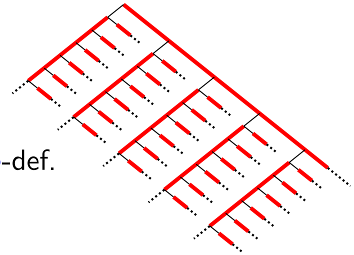

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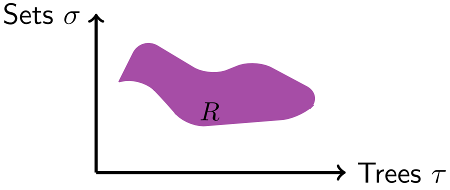

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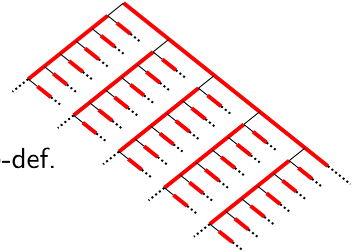

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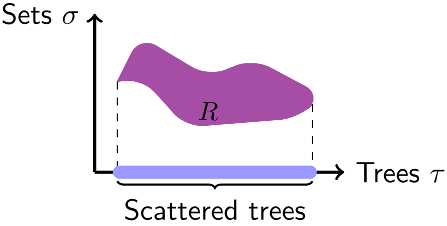

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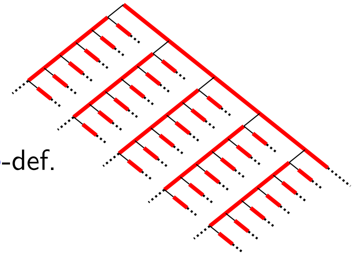

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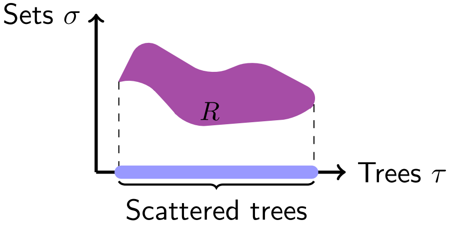

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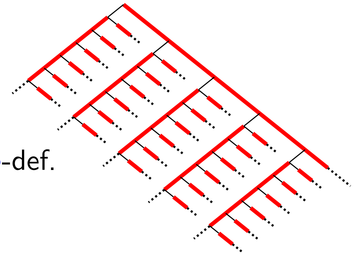

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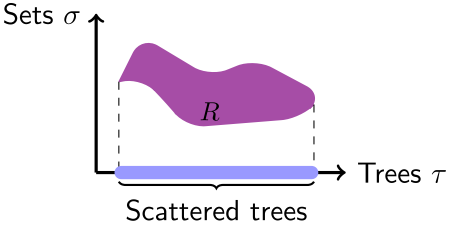

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$\leadsto$ new non-uniformisability example

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$\rightsquigarrow$ effective characterisations (weak MSO, ...)

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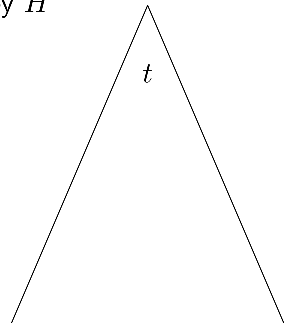

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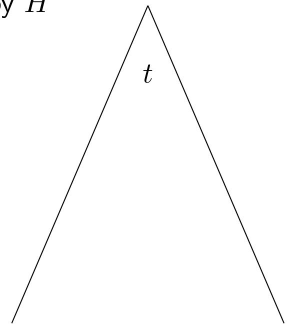

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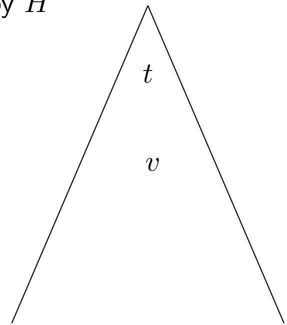

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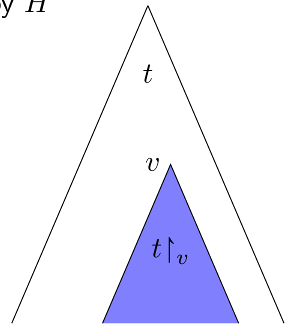

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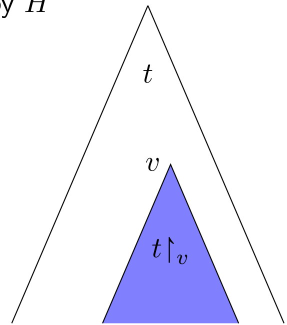

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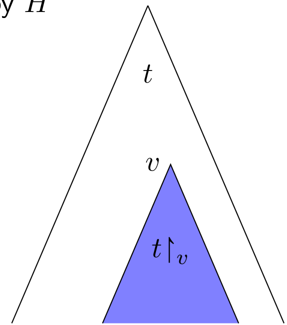
if there exists $\alpha\colon \mathrm{Trees} \to H\ldots$



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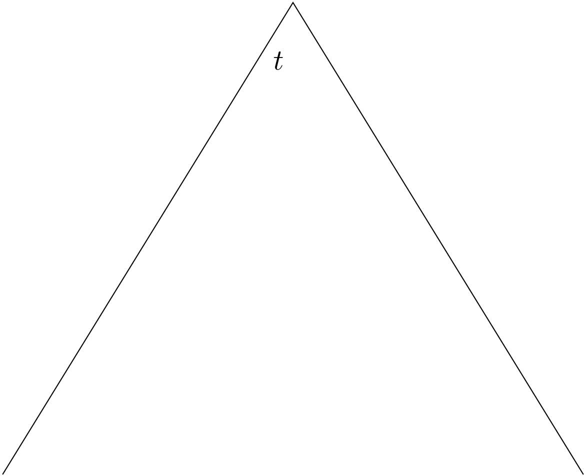

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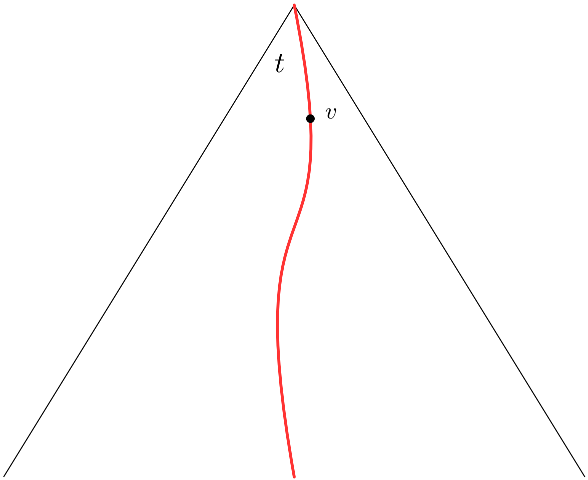

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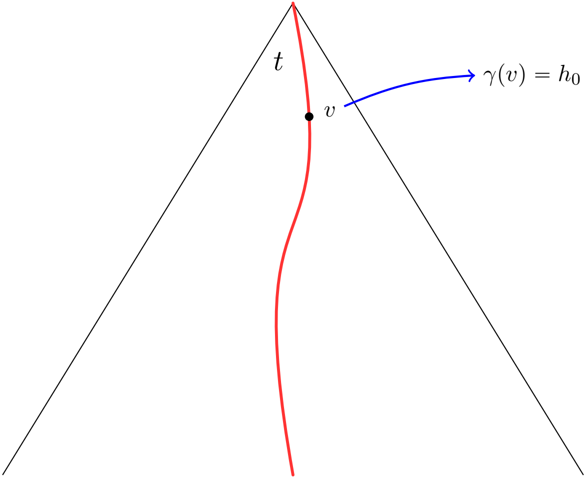

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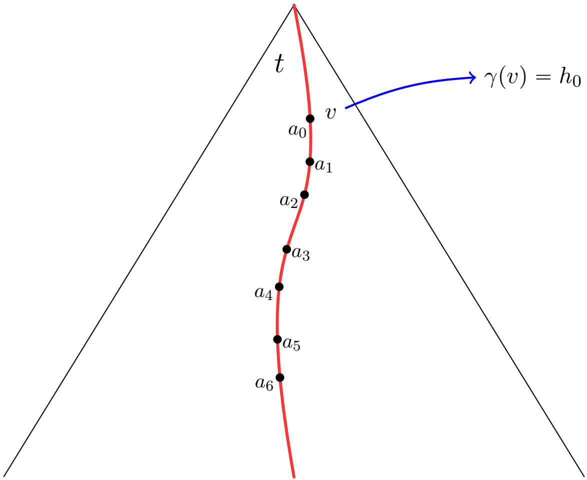

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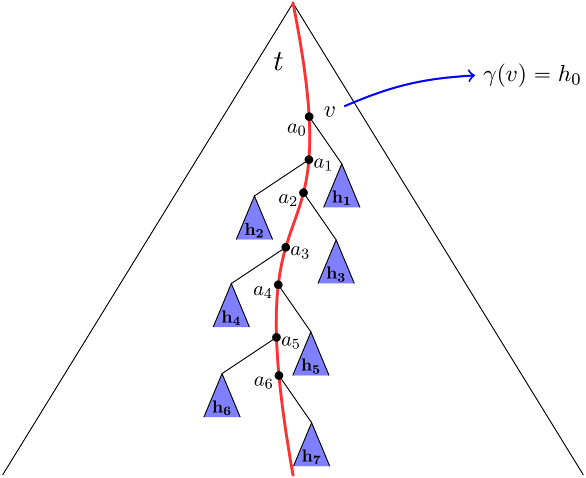

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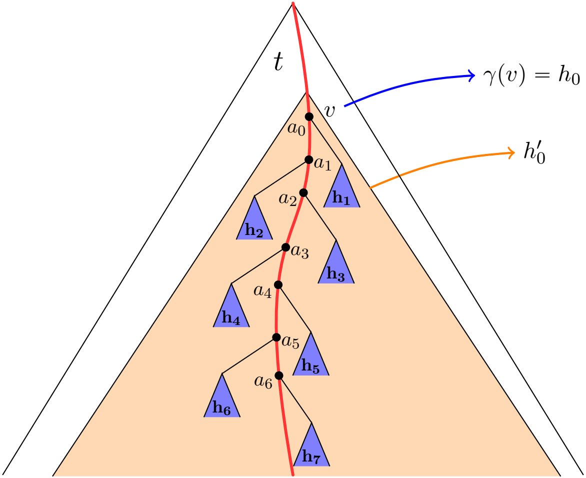

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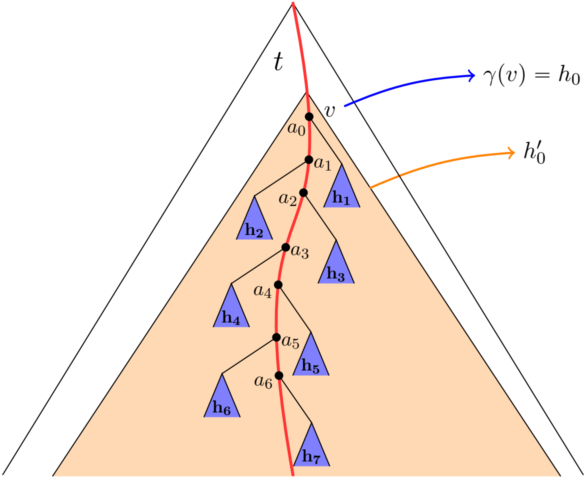

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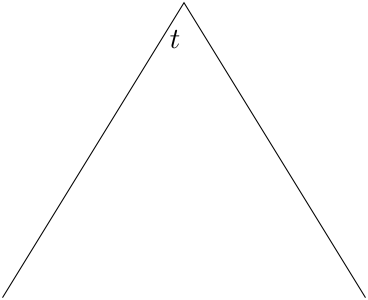

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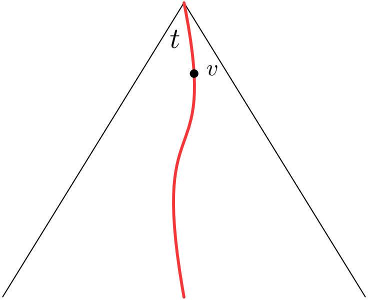

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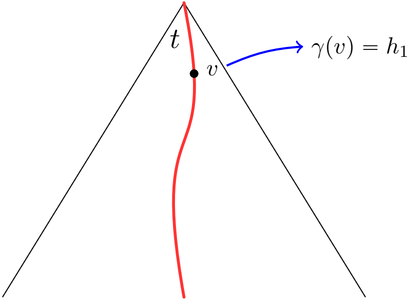

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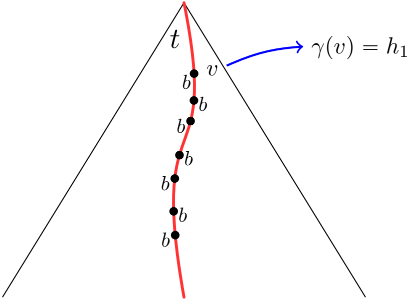

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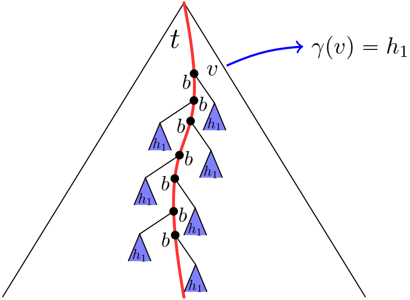

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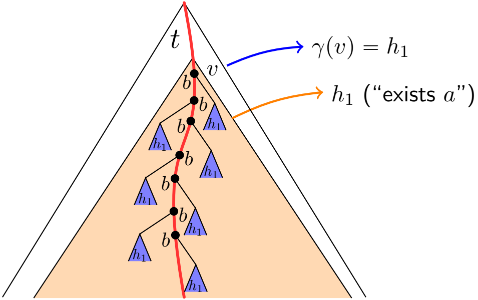

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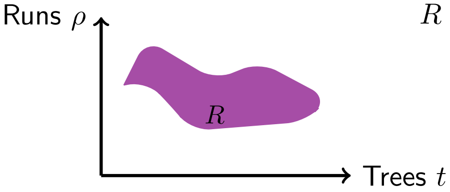

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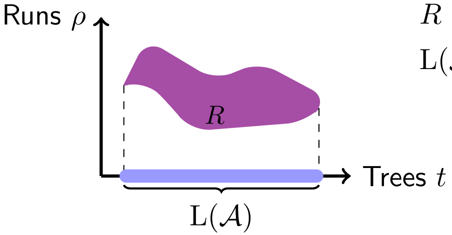

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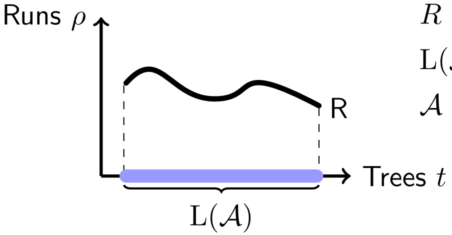

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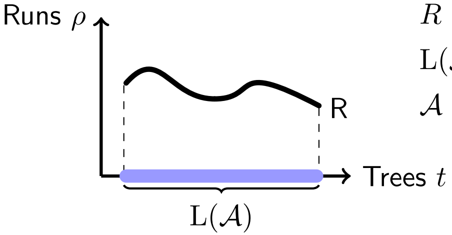

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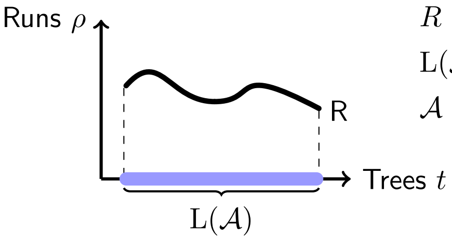

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Proof

Any unambiguous automaton for $\exists_y \ a(y)$ induces an MSO-definable choice function. ■

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Lemma (S. [2013])

If there is no MSO-def. choice function over scattered trees then

finite *prophetic* thin algebras are closed under homomorphisms.

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  - — maybe parity index bounds for unambiguous languages. . .